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ORDER, DISORDER AND PHASE TRANSITIONS IN CONDENSED MEDIA

ON THE SOLUTION OF ELECTROSTATIC PROBLEMS BY THE METHOD OF EIGENFUNCTIONS

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Abstract. The sequential scheme for solving various electrostatic problems involving a macroscopic body of arbitrary shape by the method of eigenfunctions is presented. The basic properties of eigenfunctions – regular solutions of the Laplace equation – outside, inside and on the surface of the body are considered. Under the assumption of completeness of the system of eigenfunctions on the body surface, a general solution for the electrostatic Green's function is found and the solution of Dirichlet and Neumann boundary value problems, both external and internal, is performed.

Keywords: *eigenfunctions, the Laplace equation, the Green's function, the solution of Dirichlet and Neumann boundary value problems*

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1. INTRODUCTION

The tasks of macroscopic electrostatics primarily consist of determining the potentials created by external charges and fields, influenced by the presence of dielectric and conducting bodies. For example, this includes calculating the tensor of dipole polarizability of macroscopic bodies, as well as the more complex task of the Green's function, which provides the potential created by a point charge in the presence of such bodies. Related issues include Dirichlet and Neumann boundary problems, typically included in the corresponding section of mathematical physics. Solving these tasks often requires the use of quite complex mathematical tools and is usually included in the section on Special Methods of Electrostatics. When considering each specific body of a given shape, an individual approach is used, for example, a certain system of curvilinear coordinates. However, the results obtained in this way cannot be transferred to bodies of another shape, which significantly narrows the scope of applicability of traditional methods of electrostatics. Therefore, there is interest in finding a method for solving such electrostatic problems that is applicable to bodies of arbitrary shape. Attempts to find such approaches were previously made in other works, but they turned out to be too complex and cumbersome.

In this work, a method is proposed for solving various electrostatic problems, independent of the use of specific coordinate systems. For this purpose, the eigenfunctions of the problem — regular solutions to the Laplace equation — are introduced. The main properties of these functions outside, inside the body, and on its surface are studied. The relationship between the volume eigenfunctions (outside the surface S) and their values on the surface of the body is established. The main assumption of the method presented is the completeness of the system of eigenfunctions on the interface surface S. It turns out that the system of surface eigenfunctions consists of two subsystems — values of the functions (potentials) on the surface S and their normal derivatives. Integrals over the entire surface area of the body from the product of elements of these subsystems yield the relations of orthonormality of the system of eigenfunctions. A bilinear combination of elements of these subsystems forms a completeness relation.

The properties of the eigenfunctions established in the work, along with the relations of orthonormality and completeness, allow for a consistent method of solving various electrostatic problems. The general scheme for solving such problems is as follows. The value sought $f(\mathbf{r})$, which obeys the Laplace equation, must be expressed through its value $F(\boldsymbol{\rho})$ on the surface of the body S (where $\boldsymbol{\rho}$ is the radius vector of a point belonging to S). Then, by a limiting process ($\mathbf{r} \to \boldsymbol{\rho}$), an equation for $F(\boldsymbol{\rho})$ is found. The obtained equation is solved by expanding $F(\boldsymbol{\rho})$ into a series using one of the subsystems of surface eigenfunctions. The value of $F(\boldsymbol{\rho})$ found in this way determines the sought function $f(\mathbf{r})$.

The consistent application of this action scheme allowed finding a general expression for the electrostatic Green's function. Similarly, solutions to both external and internal Dirichlet and Neumann boundary problems are provided. The problem of a body placed in a uniform electric field and its tensor of polarizability is also considered. The results obtained in these tasks are not linked to any coordinate system and are valid for bodies of arbitrary shape. Applying these results to a body of a specific shape requires determining the corresponding system of eigenfunctions. If an adequate coordinate system exists for a body of a specific shape, then the eigenfunctions and eigenvalues can be found in analytical form [9–11]. Otherwise, to determine the eigenfunctions, approximate methods, such as numerical methods, should be used.

2. SYSTEM OF EIGENFUNCTIONS

In a standard electrostatic problem, a macroscopic body of arbitrary shape with a dielectric permittivity $\varepsilon^{(i)}$ in a homogeneous medium with a dielectric permittivity $\varepsilon^{(e)}$ is typically considered. Place the origin of coordinates at the center of this body and introduce the following notations: $\mathbf{r}_e$ – the position vector of a point outside the body, $\mathbf{r}_i$ – inside the body, and $\boldsymbol{\rho}$ – the position vector of a point on the surface of the body S.

In the absence of external charges, the fundamental equations of electrostatics are:

$$\mathrm{rot}\mathbf{E} = 0, \quad \mathrm{div}\mathbf{D} = 0. \tag{2.1}$$

Here, $\mathbf{E} = -\nabla\varphi$ – represents the electric field intensity, and $\mathbf{D}$ represents the electric displacement vector. In equation (2.1),

$$\mathbf{D} = \varepsilon(\mathbf{r})\mathbf{E} = -\varepsilon(\mathbf{r})\nabla\varphi(\mathbf{r}), \tag{2.2}$$

where $\varphi(\mathbf{r})$ is the electric potential and $\varepsilon(\mathbf{r})$ is the coordinate-dependent dielectric permittivity. Let's express $\varepsilon(\mathbf{r})$ as:

$$\varepsilon(\mathbf{r}) = \varepsilon^{(e)}[1-(1-h)\theta(\mathbf{r})], \quad h = \frac{\varepsilon^{(i)}}{\varepsilon^{(e)}}, \tag{2.3}$$

where

$$\theta(\mathbf{r}_i) = 1, \quad \theta(\mathbf{r}_e) = 0.$$

Thus, the potential $\varphi(\mathbf{r})$ satisfies the equation:

$$\nabla\{[1-(1-h)\theta(\mathbf{r})]\nabla\varphi(\mathbf{r})\} = 0, \tag{2.4}$$

so that the potential both outside and inside the body obeys Laplace's equation:

$$\nabla^2\varphi^{(e)}(\mathbf{r}) = 0, \quad \nabla^2\varphi^{(i)}(\mathbf{r}) = 0, \tag{2.5}$$

where:

$$\varphi^{(e)}(\mathbf{r}) = \varphi(\mathbf{r}_e), \quad \varphi^{(i)}(\mathbf{r}) = \varphi(\mathbf{r}_i).$$

On the surface of the body (at the interface), when $\mathbf{r} = \rho$, the following conditions must be met:

$$\varphi^{(e)}(\rho) = \varphi^{(i)}(\rho), \quad \chi^{(e)}(\rho) = \mathbf{h}\chi^{(i)}(\rho). \tag{2.6}$$

Here,

$$\chi^{(e)}(\rho) = \mathbf{n}\cdot\nabla\varphi^{(e)}(\mathbf{r})\,|_{\mathbf{r}=\rho}, \tag{2.7.1}$$

$$\chi^{(i)}(\rho) = \mathbf{n}\cdot\nabla\varphi^{(i)}(\mathbf{r})\,|_{\mathbf{r}=\rho} \tag{2.7.2}$$

are the normal derivatives of the potential on the external and internal surfaces of the interface, respectively, and the vector $\mathbf{n}$ is the unit outward normal to the surface of the body.

Calculating the potential that obeys equations (2.5) with boundary conditions (2.6), under certain additional requirements for $\varphi(\mathbf{r})$ or its normal derivative, is one of the main tasks of macroscopic electrostatics. To solve this and some other tasks, let's introduce a system of eigenfunctions associated with a body of a given shape.

2.1. Polarization Functions

In the electrostatic problem of the same geometry as above, the polarization eigenfunctions $\psi_\nu(\mathbf{r})$ are solutions to Laplace's equation both outside and inside the body:

$$\nabla^2\psi_\nu^{(e)}(\mathbf{r}) = 0, \quad \nabla^2\psi_\nu^{(i)}(\mathbf{r}) = 0. \tag{2.8}$$

On the surface of the body (at the interface), these functions obey the following conditions:

$$\mathbf{r} = \rho: \quad \psi_\nu^{(e)}(\rho) = \psi_\nu^{(i)}(\rho) = \Psi_\nu(\rho), \tag{2.9}$$

$$\Phi_\nu^{(e)}(\rho) = -\varepsilon_\nu \Phi_\nu^{(i)}(\rho), \tag{2.10}$$

where

$$\Phi_\nu^{(e)}(\rho) = \mathbf{n}\cdot\nabla\psi_\nu^{(e)}(\mathbf{r})\,|_{\mathbf{r}=\rho}, \tag{2.11.1}$$

$$\Phi_\nu^{(i)}(\rho) = \mathbf{n}\cdot\nabla\psi_\nu^{(i)}(\mathbf{r})\,|_{\mathbf{r}=\rho}. \tag{2.11.2}$$

Here, as in (2.7), $\mathbf{n}$ is the unit outward normal to the surface of the body.

Inside the body, the function $\psi_\nu^{(i)}(\mathbf{r})$ is regular at $r \to 0$ and outside the body, $\psi_\nu^{(e)}(\mathbf{r})$ goes to zero at $r \to \infty$. In formula (2.10), ε_ν are the eigenvalues of the problem, which, as will be shown later, are positive.

$$\varepsilon_\nu > 0. \tag{2.12}$$

Let's integrate the second equation from (2.8) over the volume of the body $\mathbf{v}$:

$$\int_v \nabla^2\psi_\nu^{(i)}(\mathbf{r})d\mathbf{r} = \int_S \Phi_\nu^{(i)}(\rho)dS = \int \Phi_\nu^{(i)}(\rho)d\rho = 0, \tag{2.13}$$

where $dS = d\rho$ is the surface area element. The integral over $d\rho$ as well as over dS extends over the entire surface area of the body. In (2.13), when transitioning from volume integration to surface integration surrounding this volume, the Ostrogradsky–Gauss theorem is used. The same theorem is applied in the reverse operation—transitioning from surface integration to volume integration. From (2.13) it follows that:

$$\int \sigma_\nu(\rho)\mathrm{d}\rho = 0, \quad \sigma_\nu(\rho) \propto \Phi_\nu^{(e)}(\rho). \tag{2.14}$$

Here, $\sigma_\nu(\rho)$ is the surface charge density in state ν (see equation (3.30)), such that the corresponding total charge of the polarization state equals zero. Integrating the first equation from (2.8) over the volume outside the body V_e should also lead to result (2.14). In this case, the integral over V_e is transformed into integrals over the surface of the body S and over a sphere of radius R with a subsequent limit as $R \to \infty$. The integral over the sphere of infinite radius equals zero if the function $\psi_\nu^{(e)}(\mathbf{r})$ decays in the following manner:

$$r \to \infty: \quad \psi_\nu^{(e)}(\mathbf{r}) \propto \frac{1}{r^k}, \quad k \geq 2. \tag{2.15}$$

It turns out, therefore, that the polarization eigenfunctions have multipolar asymptotics.

It is easy to see that both outside and inside the body, the following equality holds:

$$\nabla\{\psi_\nu(\mathbf{r})\nabla\psi_\mu(\mathbf{r}) - \psi_\mu(\mathbf{r})\nabla\psi_\nu(\mathbf{r})\} = 0. \tag{2.16}$$

Integrating the equality (2.16) over the volume outside the body V_e, we obtain:

$$-\int\{\Psi_\nu(\rho)\Phi_\mu^{(e)}(\rho) - \Psi_\mu(\rho)\Phi_\nu^{(e)}(\rho)\}d\rho = \\ = \varepsilon_\mu\int\Psi_\nu(\rho)\Phi_\mu^{(i)}(\rho)d\rho - \\ - \varepsilon_\nu\int\Psi_\mu(\rho)\Phi_\nu^{(i)}(\rho)d\rho = 0. \tag{2.17}$$

From this, considering the equalities:

$$\int_v \nabla\psi_\nu(\mathbf{r})\nabla\psi_\mu(\mathbf{r})d\mathbf{r} = \int\Psi_\nu(\rho)\Phi_\mu^{(i)}(\rho)d\rho = \\ = \int\Psi_\mu(\rho)\Phi_\nu^{(i)}(\rho)d\rho, \tag{2.18}$$

it follows:

$$(\varepsilon_\mu - \varepsilon_\nu)\int_v \nabla\psi_\mu(\mathbf{r})\nabla\psi_\nu(\mathbf{r})d\mathbf{r} = 0. \tag{2.19}$$

Similarly, integrating the equality (2.16) over the volume of the body $\mathbf{v}$, we obtain:

$$\left(\frac{1}{\varepsilon_\mu} - \frac{1}{\varepsilon_\nu}\right)\int_{V_e} \nabla\psi_\nu(\mathbf{r})\nabla\psi_\mu(\mathbf{r})d\mathbf{r} = 0. \tag{2.20}$$

Thus, when $\varepsilon_\mu \neq \varepsilon_\nu$, the integrals of the $\nabla\psi_\nu \cdot \nabla\psi_\mu$ product over the volumes v and V_e of the body, outside the body, and over the entire space are zero. Therefore, we set:

$$\int_{V_e} (\nabla\psi_\nu \cdot \nabla\psi_\mu)d\mathbf{r} = \delta_{\nu\mu}. \tag{2.21}$$

Note that:

$$\int_{V_e} [\nabla\psi_\nu(\mathbf{r})]^2 d\mathbf{r} = -\int\Psi_\nu(\rho)\Phi_\nu^{(e)}(\rho)d\rho = \\ = \varepsilon_\nu\int\Psi_\nu(\rho)\Phi_\mu^{(i)}(\rho)d\rho,$$

So that

$$\int_{V_e} [\nabla\psi_\nu(\mathbf{r})]^2 d\mathbf{r} = \varepsilon_\nu\int_v [\nabla\psi_\nu(\mathbf{r})]^2 d\mathbf{r}. \tag{2.22}$$

From this follows, in particular, that the eigenvalues ε_ν are positive, see (2.12).

From (2.21) taking into account the equality (2.22) we find:

$$\int_v \nabla\psi_\nu(\mathbf{r})\nabla\psi_\mu(\mathbf{r})d\mathbf{r} = \frac{1}{\varepsilon_\nu}\delta_{\nu\mu} \tag{2.23}$$

or:

$$\int \Psi_\nu(\rho)\Phi_\mu^{(e)}(\rho)d\rho = -\delta_{\nu\mu}. \tag{2.24}$$

Equality (2.24) represents the orthonormality relation for the surface polarization eigenfunctions.

It should be noted that in [8, 9] a different normalization of eigenfunctions is chosen. Functions from [8, 9] (denoted here by a "tilde") are related to $\psi_\nu(\mathbf{r})$, according to (2.24), by the relation:

$$\tilde{\psi}_\nu(\mathbf{r}) = \sqrt{\frac{\varepsilon_\nu}{1+\varepsilon_\nu}}\,\psi_\nu(\mathbf{r}). \tag{2.25}$$

The same relation connects the values $\tilde{\Psi}_\nu(\rho)$, $\tilde{\Phi}_\nu^{(e)}(\rho)$ and $\tilde{\Phi}_\nu^{(i)}(\rho)$ with $\Psi_\nu(\rho)$, $\Phi_\nu^{(e)}(\rho)$ and $\Phi_\nu^{(i)}(\rho)$.

2.2. Charge functions

Laplace's equation also has a solution with monopole asymptotics, which must be taken into account for a complete understanding. In electrostatics, this behavior at $r \to \infty$ is characteristic of the potential of a charged metallic body. This potential should be chosen as an additional eigenfunction. The corresponding function $\bar{\psi}(\mathbf{r})$ (let's call it the charge function) outside the body satisfies Laplace's equation and assumes a constant value on the surface of the body:

$$\nabla^2\bar{\psi}^{(e)}(\mathbf{r}) = 0, \quad \bar{\psi}^{(e)}(\rho) = \bar{\Psi}(\rho) = \text{const}. \tag{2.26}$$

At large distances from the body, we have the following asymptotics:

$$r \to \infty: \quad \bar{\psi}^{(\mathbf{e})}(\mathbf{r}) \simeq \frac{\bar{q}}{r}, \tag{2.27}$$

where $\bar{q}$ is the charge of the body in this state. Therefore, integrating the equation for $\bar{\psi}^{(e)}(r)$ from (2.26) over the volume outside the body V_e, we obtain:

$$\bar{q} = \int\bar{\sigma}(\rho)d\rho, \quad \bar{\sigma}(\rho) = -\frac{1}{4\pi}\bar{\Phi}^{(e)}(\rho), \tag{2.28}$$

where the value of $\bar{\Phi}^{(e)}(\rho)$ is defined according to:

$$\bar{\Phi}^{(e)}(\rho) = \mathbf{n}\cdot\nabla\bar{\psi}^{(e)}(\mathbf{r})\,|_{\mathbf{r}=\rho}. \tag{2.29}$$

In (2.28), $\bar{\sigma}(\rho)$ is the charge density on the surface of the body.

Inside the metallic body, we have:

$$\bar{\psi}^{(i)}(\mathbf{r}) = \bar{\Psi}, \quad \bar{\Phi}^{(i)}(\rho) = 0, \tag{2.30}$$

so, to formally meet a boundary condition like (2.10), we should set $\bar{\varepsilon} = \infty$. We will determine the normalization of the function $\bar{\psi}(\mathbf{r})$ using a relation similar to (2.24):

$$\int\bar{\Psi}\bar{\Phi}^{(e)}(\rho)d\rho = -1. \tag{2.31}$$

Note that from (2.31), considering (2.28), we get:

$$\bar{\Psi}\bar{q} = \frac{1}{4\pi}. \tag{2.32}$$

On the other hand, there is a relation [1]:

$$\bar{q} = C\bar{\Psi}, \tag{2.33}$$

where C is the electrical capacity of the body. From here:

$$\bar{\Psi} = \frac{1}{\sqrt{4\pi C}}, \quad \bar{q} = \sqrt{\frac{C}{4\pi}}. \tag{2.34}$$

Note that according to (2.14):

$$\int\bar{\Psi}\Phi_\nu^{(e)}(\rho)d\rho = \bar{\Psi}\int\Phi_\nu^{(e)}(\rho)d\rho = 0. \tag{2.35}$$

Next, we integrate the equation:

$$\nabla\{\bar{\psi}(\mathbf{r})\nabla\psi_\nu(\mathbf{r}) - \psi_\nu(\mathbf{r})\nabla\bar{\psi}(\mathbf{r})\} = 0 \tag{2.36}$$

over the volume outside the body:

$$\int\bar{\Psi}\Phi_\nu^{(e)}(\rho)d\rho - \int\Psi_\nu(\rho)\bar{\Phi}^{(e)}(\rho)d\rho = 0. \tag{2.37}$$

According to (2.35), the first integral in (2.37) equals zero, so:

$$\int\Psi_\nu(\rho)\bar{\Phi}^{(e)}(\rho)d\rho = 0. \tag{2.38}$$

Thus, according to (2.24), (2.31), (2.35), and (2.38), the system of surface eigenfunctions is orthonormalized on the surface of the body.

It is known that in a two-dimensional problem, the monopole potential far from a charged body increases logarithmically without bound. In this case, the charge function $\bar{\psi}^{(e)}(\mathbf{r})$ at has the $r \to \infty$ following asymptotics: $\bar{\psi}^{(e)}(\mathbf{r}) \sim \ln r$. Such a function is not normalizable over the entire infinite plane.

Therefore, when studying two-dimensional charge states, it is necessary to consider this problem in a region bounded by a circle of large, but finite radius R ($R \gg a$, where a is a characteristic size of the body), and it is required that the $\bar{\psi}^{(e)}(\mathbf{r})$ function equals zero at $r = R$.

It is assumed that the system of surface eigenfunctions is complete. In this case, any arbitrary function $f(\rho)$ can be expanded into the following series:

$$f(\rho) = \sum_{\nu} f_{\nu}\Psi_{\nu}(\rho) + \bar{f}\bar{\Psi}(\rho). \quad (2.39)$$

Considering (2.24), (2.31), (2.35), and (2.38), from the expansion (2.39) we find the coefficients f_{ν} and $\bar{f}$:

$$f_{\nu} = -\int f(\rho)\Phi_{\nu}^{(e)}(\rho)d\rho, \quad (2.40.1)$$

$$\bar{f} = -\int f(\rho)\bar{\Phi}^{(e)}(\rho)d\rho. \quad (2.40.2)$$

By substituting coefficients f_{ν} and $\bar{f}$ into (2.39) and requiring that this expansion converges to the function $f(\rho)$ itself, we arrive at the equality:

$$\sum_{\nu}\Psi_{\nu}(\rho)\Phi_{\nu}^{(e)}(\rho') + \bar{\Psi}(\rho)\bar{\Phi}^{(\mathbf{e})}(\rho') = -\delta(\rho - \rho'), \quad (2.41)$$

which is a completeness relation. A similar expansion of the function $f(\rho)$ into a series by the subsystem $\{\Phi_{\nu}^{(e)}(\rho), \bar{\Phi}^{(e)}(\rho)\}$ also leads to a completeness relation of the form (2.41).

3. RELATIONSHIP BETWEEN VOLUME AND SURFACE FUNCTIONS

Let's introduce the zeroth Green's function:

$$G_0(\mathbf{r} - \mathbf{r}') = -\frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|}, \quad (3.1)$$

obeying the equation:

$$\nabla_{\mathbf{r}'}^2 G_0(\mathbf{r} - \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'). \quad (3.2)$$

It is easy to see that both outside and inside the body the following equality holds:

$$\psi_{\nu}(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}') = \\ = \nabla_{\mathbf{r}'}\{\psi_{\nu}(\mathbf{r}')\nabla_{\mathbf{r}'}G_0(\mathbf{r} - \mathbf{r}') - G_0(\mathbf{r} - \mathbf{r}')\nabla_{\mathbf{r}'}\psi_{\nu}(\mathbf{r}')\}. \quad (3.3)$$

Let us multiply equation (3.3) by $d\mathbf{r}'$ and integrate it over the volume V_e outside the body:

$$\psi_{\nu}(\mathbf{r})[1 - \theta(\mathbf{r})] = \\ = -\int k(\mathbf{r}, \rho')\Psi_{\nu}(\rho')d\rho' + \int G_0(\mathbf{r} - \rho')\Phi_{\nu}^{(e)}(\rho')d\rho'. \quad (3.4)$$

Here:

$$\theta(\mathbf{r}_e) = 0, \quad \theta(\mathbf{r}_i) = 1$$

and:

$$k(\mathbf{r}, \rho') = \mathbf{n} \cdot \nabla_{\mathbf{r}'}G_0(\mathbf{r} - \mathbf{r}')|_{\mathbf{r}'=\rho'}. \quad (3.5)$$

Let us integrate equation (3.2) over the volume of the body:

$$\int_{v}\nabla_{\mathbf{r}'}^2 G_0(\mathbf{r} - \mathbf{r}')d\mathbf{r}' = \int k(\mathbf{r}, \rho')d\rho' = \theta(\mathbf{r}). \quad (3.6)$$

From this:

$$\int k(\mathbf{r}_e, \rho')d\rho' = 0, \quad (3.7)$$

$$\int k(\mathbf{r}_i, \rho')d\rho' = 1. \quad (3.8)$$

From relation (3.4) at $\mathbf{r} = \mathbf{r}_e$ and $\mathbf{r} = \mathbf{r}_i$, we respectively find:

$$\psi_{\nu}^{(e)}(\mathbf{r}) = \\ = -\int k(\mathbf{r}_e, \rho')\Psi_{\nu}(\rho')d\rho' + \\ + \int G_0(\mathbf{r_e} - \rho')\Phi_{\nu}^{(e)}(\rho')d\rho', \quad (3.9)$$

$$\int k(\mathbf{r}_i, \rho')\Psi_{\nu}(\rho')d\rho' - \\ - \int G_0(\mathbf{r}_i - \rho')\Phi_{\nu}^{(e)}(\rho')d\rho' = 0. \quad (3.10)$$

On the other hand, integrating (3.3) over the volume of the body v yields:

$$\psi_{\nu}(\mathbf{r})\theta(\mathbf{r}) = \\ = \int k(\mathbf{r}, \rho')\Psi_{\nu}(\rho')d\rho' + \\ + \frac{1}{\varepsilon_{\nu}}\int G_0(\mathbf{r} - \rho')\Phi_{\nu}^{(e)}(\rho')d\rho'. \quad (3.11)$$

From here at $\mathbf{r} = \mathbf{r}_i$ and $\mathbf{r} = \mathbf{r}_e$ respectively, we find:

$$\psi_\nu^{(i)}(\mathbf{r}) = \\ = \int k(\mathbf{r}_i,\rho')\Psi_\nu(\rho')d\rho' + \\ + \frac{1}{\varepsilon_\nu}\int G_0(\mathbf{r}_i - \rho')\Phi_\nu^{(e)}(\rho')d\rho', \quad (3.12)$$

$$\int k(\mathbf{r}_e,\rho')\Psi_\nu(\rho')d\rho' + \\ + \frac{1}{\varepsilon_\nu}\int G_0(\mathbf{r}_e - \rho')\Phi_\nu^{(e)}(\rho')d\rho' = 0. \quad (3.13)$$

From the equality (3.9) considering (3.13), and from (3.12) considering (3.10) we get:

$$\psi_\nu^{(e)}(\mathbf{r}) = -(1+\varepsilon_\nu)\int k(\mathbf{r}_e,\rho')\Psi_\nu(\rho')d\rho', \quad (3.14)$$

$$\psi_\nu^{(i)}(\mathbf{r}) = \frac{1+\varepsilon_\nu}{\varepsilon_\nu}\int k(\mathbf{r}_i,\rho')\Psi_\nu(\rho')d\rho'. \quad (3.15)$$

From (3.14) at $\mathbf{r}_e \to \rho$ and from (3.15) at $\mathbf{r}_i \to \rho$, we have:

$$\Psi_\nu(\rho) = -(1+\varepsilon_\nu)\int K^{(e)}(\rho,\rho')\Psi_\nu(\rho')d\rho', \quad (3.16)$$

$$\Psi_\nu(\rho) = \frac{1+\varepsilon_\nu}{\varepsilon_\nu}\int K^{(i)}(\rho,\rho')\Psi_\nu(\rho')d\rho'. \quad (3.17)$$

Here:

$$K^{(e)}(\rho,\rho') = \lim_{\mathbf{r}_e\to\rho} k(\mathbf{r}_e,\rho'), \quad (3.18.1)$$

$$K^{(\mathbf{i})}(\rho,\rho') = \lim_{\mathbf{r}_i\to\rho} k(\mathbf{r}_i,\rho'). \quad (3.18.2)$$

The values (3.16) and (3.17) must be equal, which is possible only if the following equality is met:

$$K^{(e)}(\rho,\rho') - K^{(i)}(\rho,\rho') = -\delta(\rho-\rho'). \quad (3.19)$$

Similarly, from (3.9) and (3.12), considering relations (3.10) and (3.13), we find:

$$\psi_\nu^{(e)}(\mathbf{r}) = \frac{1+\varepsilon_\nu}{\varepsilon_\nu}\int G_0(\mathbf{r}_e - \rho')\Phi_\nu^{(e)}(\rho')d\rho', \quad (3.20)$$

$$\psi_\nu^{(i)}(\mathbf{r}) = \frac{1+\varepsilon_\nu}{\varepsilon_\nu}\int G_0(\mathbf{r}_i - \rho')\Phi_\nu^{(e)}(\rho')d\rho'. \quad (3.21)$$

Integrating the equality:

$$\bar\psi(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}') = \\ = \nabla_{r'}\{\bar\psi(\mathbf{r}')\nabla_{r'}G_0(\mathbf{r}-\mathbf{r}') - G_0(\mathbf{r}-\mathbf{r}')\nabla_{r'}\bar\psi(\mathbf{r}')\} \quad (3.22)$$

over the volume V_e outside the body, at $\mathbf{r} = \mathbf{r}_e$ and $\mathbf{r} = \mathbf{r}_i$ respectively, we obtain:

$$\bar\psi^{(e)}(\mathbf{r}) = \int G_0(\mathbf{r}_e - \rho')\bar\Phi^{(e)}(\rho')d\rho', \quad (3.23)$$

$$\bar\Psi = \int G_0(\mathbf{r}_i - \rho')\bar\Phi^{(e)}(\rho')d\rho'. \quad (3.24)$$

In deriving formulas (3.23) and (3.24), equalities (3.7) and (3.8) were taken into account.

Using the expansion

$$r\to\infty: \ G_0(\mathbf{r}-\mathbf{r}') \simeq -\frac{1}{4\pi}\left\{\frac{1}{r} + \frac{\mathbf{r}\mathbf{r}'}{r^3} + \cdots\right\}, \quad (3.25)$$

from (3.23) we find the asymptotics of the charge function:

$$r\to\infty: \ \bar\psi^{(e)}(\mathbf{r}) \simeq \frac{\bar q}{r} + \frac{\mathbf{r}\bar{\mathbf{d}}}{r^3} + \cdots, \quad (3.26)$$

where $\bar q$ is the total charge, see (2.28), $\bar{\mathbf{d}}$ is the dipole moment of this state,

$$\bar{\mathbf{d}} = \int \rho\,\bar\sigma(\rho)d\rho. \quad (3.27)$$

Here, $\bar\sigma(\rho)$ is the surface charge density defined in (2.28). Correspondingly, from (3.20) we find the asymptotics of the polarization function:

$$r\to\infty: \ \psi_\nu^{(e)}(\mathbf{r}) \simeq \frac{\mathbf{r}\mathbf{d}_\nu}{r^3} + \cdots, \quad (3.28)$$

where $\mathbf{d}_\nu$ is the dipole moment of the state ν

$$\mathbf{d}_\nu = \int \rho\,\sigma_\nu(\rho)d\rho \quad (3.29)$$

and $\sigma_\nu(\rho)$ is the surface charge density of this state:

$$\sigma_\nu(\rho) = -\frac{1}{4\pi}\Phi_\nu^{(e)}(\rho)\frac{1+\varepsilon_\nu}{\varepsilon_\nu}. \quad (3.30)$$

Let us multiply formula (3.20) by $(\varepsilon_\nu/(1+\varepsilon_\nu))\Psi_\nu(\rho)$ and sum over ν. Adding $\Psi_\nu(\rho)\Phi_\nu^{(e)}(\rho')$ and subtracting the term $\bar\Psi\bar\Phi^{(e)}(\rho')$ we obtain, considering the completeness relation (2.41):

$$G_0(\mathbf{r}_e - \rho') = \\ = -\left\{\sum_\nu \frac{\varepsilon_\nu}{1+\varepsilon_\nu}\psi_\nu^{(e)}(\mathbf{r})\Psi_\nu(\rho') + \bar\psi^{(e)}(\mathbf{r})\bar\Psi\right\}. \quad (3.31)$$

Similarly, from (3.21) we find:

$$G_0(\mathbf{r}_i - \rho') =$$
$$= -\left\{\sum_\nu \frac{\varepsilon_\nu}{1+\varepsilon_\nu}\psi_\nu^{(i)}(\mathbf{r})\Psi_\nu(\rho') + \bar{\Psi}^2\right\}. \quad (3.32)$$

Considering the equation (3.2) for the zeroth Green's function, we have:

$$G_0(\mathbf{r}_e - \mathbf{r}_i')\delta(\mathbf{r}_i' - \mathbf{r}'') =$$
$$= \nabla_{\mathbf{r}''}\{G_0(\mathbf{r}_e - \mathbf{r}'')\nabla_{\mathbf{r}''}G_0(\mathbf{r}_i' - \mathbf{r}'') -$$
$$-G_0(\mathbf{r}_i' - \mathbf{r}'')\nabla_{\mathbf{r}''}G_0(\mathbf{r}_e - \mathbf{r}'')\} +$$
$$+G_0(\mathbf{r}_e - \mathbf{r}_i')\delta(\mathbf{r}_e - \mathbf{r}''). \quad (3.33)$$

Integrate equality (3.33) over the volume of the body v:

$$G_0(\mathbf{r}_e - \mathbf{r}_i') =$$
$$= \int k(\mathbf{r}_i', \rho'')G_0(\mathbf{r}_e - \rho'')d\rho'' -$$
$$-\int k(\mathbf{r}_e, \rho'')G_0(\mathbf{r}_i' - \rho'')d\rho''. \quad (3.34)$$

By substituting formula (3.34) into expressions (3.31), (3.32) and using relations (3.14), (3.15), we find:

$$G_0(\mathbf{r}_e - \mathbf{r}_i') =$$
$$= -\left\{\sum_\nu \frac{\varepsilon_\nu}{1+\varepsilon_\nu}\psi_\nu^{(e)}(\mathbf{r})\psi_\nu^{(i)}(\mathbf{r}') + \bar{\psi}^{(e)}(\mathbf{r})\bar{\psi}^{(i)}(\mathbf{r}')\right\}, \quad (3.35)$$

where $\bar{\psi}^{(i)}(\mathbf{r}') = \bar{\Psi} = \text{const}$. In deriving formula (3.35), relations (3.23) and (3.24) were also considered.

Using expression (3.35) for $G_0(\mathbf{r}_e - \mathbf{r}'_i)$ we find values of $k(\mathbf{r}_e, \rho')$ and $k(\mathbf{r}_i, \rho')$, defined according to (3.5):

$$k(\mathbf{r}_e, \rho') = \sum_\nu \frac{1}{1+\varepsilon_\nu}\psi_\nu^{(e)}(\mathbf{r})\Phi_\nu^{(e)}(\rho'), \quad (3.36)$$

$$k(\mathbf{r}_i, \rho') =$$
$$= -\left\{\sum_\nu \frac{\varepsilon_\nu}{1+\varepsilon_\nu}\psi_\nu^{(i)}(\mathbf{r})\Phi_\nu^{(e)}(\rho') + \bar{\psi}^{(i)}(\mathbf{r})\bar{\Phi}^{(e)}(\rho')\right\}. \quad (3.37)$$

Here also $\bar{\psi}^{(i)}(\mathbf{r}) = \bar{\Psi} = \text{const}$. From this, at $\mathbf{r}_e \to \rho$ and $\mathbf{r}_i \to \rho$ respectively, we find:

$$K^{(e)}(\rho, \rho') = \sum_\nu \frac{1}{1+\varepsilon_\nu}\Psi_\nu(\rho)\Phi_\nu^{(e)}(\rho'), \quad (3.38)$$

$$K^{(i)}(\rho, \rho') =$$
$$= -\left\{\sum_\nu \frac{\varepsilon_\nu}{1+\varepsilon_\nu}\Psi_\nu(\rho)\Phi_\nu^{(e)}(\rho') + \bar{\Psi}(\rho)\bar{\Phi}^{(e)}(\rho')\right\}. \quad (3.39)$$

The substitution of formulas (3.38) and (3.39) into relation (3.19) turns it, due to the completeness relation (2.41), into an identity.

4. GREEN'S FUNCTION

If there are external charges in the problem formulated in Section 2, the main equation of electrostatics takes the form:

$$\text{div}\mathbf{D} = 4\pi\rho, \quad (4.1)$$

where $\rho(\mathbf{r})$ is the volume density of these charges. In this case, the potential $\varphi(\mathbf{r})$ obeys the equation:

$$\nabla\{[1-(1-h)\theta(\mathbf{r})]\nabla\varphi(\mathbf{r})\} = -\frac{1}{\varepsilon^{(e)}}4\pi\rho, \quad (4.2)$$

where, as in (2.3), $h = \varepsilon^{(i)} / \varepsilon^{(e)}$. The solution of equation (4.2), which gives the potential induced by external charges, is written in the form:

$$\varphi(\mathbf{r}) = -\frac{4\pi}{\varepsilon^{(e)}}\int G(\mathbf{r}, \mathbf{r}')\rho(\mathbf{r}')d\mathbf{r}'. \quad (4.3)$$

Here $G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r}', \mathbf{r})$ is the electrostatic Green's function, satisfying the equation:

$$\nabla_{r'}\{[1-(1-h)\theta(\mathbf{r}')]\nabla_{\mathbf{r}'}G(\mathbf{r}, \mathbf{r}')\} = \delta(\mathbf{r} - \mathbf{r}'). \quad (4.4)$$

It follows that at $\mathbf{r}' = \mathbf{r}_e'$ and $\mathbf{r}' = \mathbf{r}_i'$ we have:

$$\mathbf{r}' = \mathbf{r}_e': \quad \nabla_{\mathbf{r}'}^2 G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'), \quad (4.5)$$

$$\mathbf{r}' = \mathbf{r}_i': \quad \nabla_{\mathbf{r}'}^2 G(\mathbf{r}, \mathbf{r}') = \frac{1}{h}\delta(\mathbf{r} - \mathbf{r}'). \quad (4.6)$$

Green's function obeys the following boundary conditions:

$$G(\mathbf{r}, \mathbf{r}_e')|_{\mathbf{r}_e' \to \rho'} = G(\mathbf{r}, \mathbf{r}_i')|_{\mathbf{r}_i' \to \rho'}, \quad (4.7)$$

$$j^{(e)}(\mathbf{r}, \rho') = hj^{(i)}(\mathbf{r}, \rho'). \quad (4.8)$$

Here:

$$\mathbf{j}^{(e)}(\mathbf{r}, \rho') = \mathbf{n}\cdot\nabla_{\mathbf{r}'}\mathbf{G}(\mathbf{r}, \mathbf{r}_e')|_{\mathbf{r}_e' \to \rho'}, \quad (4.9)$$

$$j^{(i)}(\mathbf{r}, \rho') = \mathbf{n}\cdot\nabla_{\mathbf{r}'}G(\mathbf{r}, \mathbf{r}_i')|_{\mathbf{r}_i' \to \rho'}. \quad (4.10)$$

Integrating equation (4.9) over the volume V_e outside the body and (4.10) over the volume of the body v, we obtain:

$$\int j^{(e)}(\mathbf{r}_e,\rho)d\rho' = 0, \int j^{(e)}(\mathbf{r}_i,\rho')d\rho' = 1; \quad (4.11.1)$$

$$\int j^{(i)}(\mathbf{r}_i,\rho')d\rho' = \frac{1}{h}, \int j^{(i)}(\mathbf{r}_e,\rho')d\rho' = 0. \quad (4.11.2)$$

Here, it is assumed that $G(\mathbf{r},\mathbf{r}')$ has the same asymptotics as $G_0(\mathbf{r}-\mathbf{r}')$:

$$r \to \infty: \quad G(\mathbf{r},\mathbf{r}') \simeq -\frac{1}{4\pi r}. \quad (4.12)$$

This assumption will be directly confirmed below.

Considering equations (3.2), (4.5), and (4.6), we have the following equalities:

$$\begin{gathered}\mathbf{r}'' = \mathbf{r}''_e: \quad G(\mathbf{r},\mathbf{r}')\delta(\mathbf{r}'-\mathbf{r}'') = \\ = \nabla_{\mathbf{r}''}\{G(\mathbf{r},\mathbf{r}'')\nabla_{\mathbf{r}''}G_0(\mathbf{r}'-\mathbf{r}'') - \\ -G_0(\mathbf{r}'-\mathbf{r}'')\nabla_{\mathbf{r}''}G(\mathbf{r},\mathbf{r}'')\} + \\ +G_0(\mathbf{r}'-\mathbf{r})\delta(\mathbf{r}-\mathbf{r}''), \end{gathered} \quad (4.13)$$

$$\begin{gathered}\mathbf{r}'' = \mathbf{r}''_\mathbf{i}: \quad G(\mathbf{r},\mathbf{r}')\delta(\mathbf{r}'-\mathbf{r}'') = \\ = \nabla_{\mathbf{r}''}\{G(\mathbf{r},\mathbf{r}'')\nabla_{\mathbf{r}''}G_0(\mathbf{r}'-\mathbf{r}'') - \\ -G_0(\mathbf{r}'-\mathbf{r}'')\nabla_{\mathbf{r}''}G(\mathbf{r},\mathbf{r}'')\} + \\ +\frac{1}{h}G_0(\mathbf{r}'-\mathbf{r})\delta(\mathbf{r}-\mathbf{r}''). \end{gathered} \quad (4.14)$$

Let us assume that in (4.13), $\mathbf{r} = \mathbf{r}_e$, $\mathbf{r}' = \mathbf{r}'_e$, and integrate the obtained equality over the volume V_e outside the body:

$$\begin{gathered}G(\mathbf{r}_e,\mathbf{r}'_e) = \\ = -\int k(\mathbf{r}'_e,\rho'')G(\mathbf{r}_e,\rho'')d\rho'' + \\ +\int j^{(e)}(\mathbf{r}_e,\rho'')G_0(\mathbf{r}'_e-\rho'')d\rho'' + G_0(\mathbf{r}_e-\mathbf{r}'_e). \end{gathered} \quad (4.15)$$

Similarly, at $\mathbf{r} = \mathbf{r}_e$, $\mathbf{r}' = \mathbf{r}'_e$ integrating the equality (4.14) over the volume of the body v gives:

$$\begin{gathered}\int k(\mathbf{r}'_e,\rho'')G(\mathbf{r}_e,\rho'')d\rho'' - \\ -\frac{1}{h}\int j^{(e)}(\mathbf{r}_e,\rho'')G_0(\mathbf{r}'_e-\rho'')d\rho'' = 0. \end{gathered} \quad (4.16)$$

Excluding from (4.15) and (4.16) the integral containing $j^{(e)}(\mathbf{r}_e,\rho'')$, we find:

$$\begin{gathered}G(\mathbf{r}_e,\mathbf{r}'_e) = \\ = -(1-h)\int k(\mathbf{r}'_e,\rho'')G(\mathbf{r}_e,\rho'')d\rho'' + G_0(\mathbf{r}_\mathbf{e}-\mathbf{r}'_e). \end{gathered} \quad (4.17)$$

Setting in (4.17) $\mathbf{r}'_e = \rho'$, we get an equation for the value of $G(\mathbf{r}_e,\rho')$:

$$\begin{gathered}G(\mathbf{r}_e,\rho') = \\ = -(1-h)\int K^{(e)}(\rho',\rho'')G(\mathbf{r}_e,\rho'')d\rho'' + G_0(\mathbf{r}_e-\rho') \end{gathered} \quad (4.18)$$

with $K^{(e)}(\rho',\rho'')$, defined according to (3.18). The value of $G(\mathbf{r}_e,\rho')$ is sought as an expansion over the system of surface eigenfunctions:

$$G(\mathbf{r}_e,\rho') = \sum_\nu a_\nu \Psi_\nu(\rho') + \bar{a}\bar{\Psi}. \quad (4.19)$$

Substituting (4.19) into (4.18) considering (3.7) and (3.16) gives:

$$\begin{gathered}\sum_\nu a_\nu \Psi_\nu(\rho') + \bar{a}\bar{\Psi} = \\ = \sum_\nu a_\nu \frac{1-h}{1+\varepsilon_\nu}\Psi_\nu(\rho') + G_0(\mathbf{r}_e-\rho'). \end{gathered} \quad (4.20)$$

Changing in (4.20) the index $\nu \to \mu$, let us multiply the obtained equality by $\Phi_\nu^{(e)}(\rho')$, then integrate over ρ' and find the coefficient a_ν. Then, multiplying (4.20) by $\bar{\Phi}^{(e)}(\rho')$ and integrating over ρ' we find the coefficient $\bar{a}$. Thus,

$$a_\nu = -\frac{\varepsilon_\nu}{h+\varepsilon_\nu}\psi_\nu^{(e)}(\mathbf{r}), \quad \bar{a} = -\bar{\psi}^{(e)}(\mathbf{r}), \quad (4.21)$$

so that:

$$\begin{gathered}G(\mathbf{r}_e,\rho') = \\ = -\sum_\nu \frac{\varepsilon_\nu}{h+\varepsilon_\nu}\psi_\nu^{(e)}(\mathbf{r})\Psi_\nu(\rho') - \bar{\psi}^{(e)}(\mathbf{r})\bar{\Psi}. \end{gathered} \quad (4.22)$$

In deriving expressions for the coefficients (4.21), formulas (3.20) and (3.23), as well as the orthonormality relation (2.24), (2.31), (2.35), (2.38) were used. Substitution of (4.22) into (4.17) gives:

$$\begin{gathered}G(\mathbf{r}_e,\mathbf{r}'_e) = G_0(\mathbf{r}_e-\mathbf{r}'_e) - \\ -\sum_\nu \frac{1-h}{h+\varepsilon_\nu}\frac{\varepsilon_\nu}{1+\varepsilon_\nu}\psi_\nu^{(e)}(\mathbf{r})\psi_\nu^{(e)}(\mathbf{r}'). \end{gathered} \quad (4.23)$$

Now set in (4.13), (4.14):

$$\mathbf{r} = \mathbf{r}_e, \quad \mathbf{r}' = \mathbf{r}'_e$$

and integrate (4.13) over the volume V_e outside the body, and (4.14) — over the volume of the body v:

$$\begin{gathered}G(\mathbf{r}_i,\mathbf{r}'_e) = -\int k(\mathbf{r}'_e,\rho'')G(\mathbf{r}_i,\rho'')d\rho'' + \\ +\int j^{(e)}(\mathbf{r}_i,\rho'')G_0(\mathbf{r}'_e-\rho'')d\rho'', \end{gathered} \quad (4.24)$$

$$\int k(\mathbf{r}'_e,\rho'')G(\mathbf{r}_i,\rho'')d\rho'' - \\ -\frac{1}{h}\int j^{(e)}(\mathbf{r}_i,\rho'')G_0(\mathbf{r}'_e-\rho'')d\rho'' + \frac{1}{h}G_0(\mathbf{r}_i-\mathbf{r}'_e) = 0. \tag{4.25}$$

Excluding from (4.24), (4.25), as usual, the integral with $j^{(e)}(\mathbf{r_i},\rho'')$ we obtain:

$$G(\mathbf{r}_i,\mathbf{r}'_e) = \\ = -(1-h)\int k(\mathbf{r}'_e,\rho'')G(\mathbf{r}_i,\rho'')d\rho'' + G_0(\mathbf{r}_i-\mathbf{r}'_e). \tag{4.26}$$

Setting in (4.26) $\mathbf{r}'_e = \rho'$, we find an equation for the value of $G(\mathbf{r}_i,\rho')$. Solving the obtained equation similarly as (4.18), we find for $G(\mathbf{r}_i,\rho')$ an expression different from (4.22) only by replacing $\mathbf{r}_e$ with $\mathbf{r}_i$. Therefore, in general:

$$G(\mathbf{r},\rho') = \\ = -\left\{\sum_\nu \frac{\varepsilon_\nu}{h+\varepsilon_\nu}\psi_\nu(\mathbf{r})\Psi_\nu(\rho') + \bar{\psi}(\mathbf{r})\bar{\Psi}\right\}. \tag{4.27}$$

Substituting (4.27) at $\mathbf{r} = \mathbf{r}_i$ in formula (4.26) gives:

$$G(\mathbf{r}_i,\mathbf{r}'_e) = \\ = -\left\{\sum_\nu \frac{\varepsilon_\nu}{h+\varepsilon_\nu}\psi_\nu^{(i)}(\mathbf{r})\psi_\nu^{(e)}(\mathbf{r}') + \bar{\psi}^{(i)}(\mathbf{r})\bar{\psi}^{(e)}(\mathbf{r}')\right\} \tag{4.28}$$

with $\bar{\psi}^{(i)}(\mathbf{r}) = \bar{\Psi}$. In deriving (4.28), the expansion (3.35) for $G_0(\mathbf{r}_e - \mathbf{r}'_i)$ was used. Due to the symmetry of the Green's function from (4.28), it follows:

$$G(\mathbf{r}_e,\mathbf{r}'_i) = \\ = -\left\{\sum_\nu \frac{\varepsilon_\nu}{h+\varepsilon_\nu}\psi_\nu^{(e)}(\mathbf{r})\psi_\nu^{(i)}(\mathbf{r}') + \bar{\psi}^{(e)}(\mathbf{r})\bar{\psi}^{(i)}(\mathbf{r}')\right\}. \tag{4.29}$$

Finally, setting in (4.13), (4.14) $\mathbf{r} = \mathbf{r}_i$, $\mathbf{r}' = \mathbf{r}'_i$, and integrating the obtained equalities, as usual, over V_e and v, we get:

$$-\int k(\mathbf{r}'_i,\rho'')G(\mathbf{r}_i,\rho'')d\rho'' + \\ +\int j^{(e)}(\mathbf{r}_i,\rho'')G_0(\mathbf{r}'_i-\rho'')d\rho'' = 0, \tag{4.30}$$

$$G(\mathbf{r}_i,\mathbf{r}'_i) = \int k(\mathbf{r}'_i,\rho'')G(\mathbf{r}_i,\rho'')d\rho'' - \\ -\frac{1}{h}\int j^{(e)}(\mathbf{r}_i,\rho'')G_0(\mathbf{r}'_i-\rho'')d\rho'' + G_0(\mathbf{r}_i-\mathbf{r}'_i). \tag{4.31}$$

Excluding from (4.30) and (4.31) the integral containing $j^{(e)}(\mathbf{r}_i,\rho'')$ we find:

$$G(\mathbf{r}_i,\mathbf{r}'_i) = -\frac{1-h}{h}\int k(\mathbf{r}'_i,\rho'')G(\mathbf{r}_i,\rho'')d\rho'' + \\ +\frac{1}{h}G_0(\mathbf{r}_i-\mathbf{r}'_i). \tag{4.32}$$

Setting in (4.32) $\mathbf{r}'_i = \rho'$, we obtain an equation for the value of $G(\mathbf{r}_i,\rho')$, the solution to which is given by formula (4.27) at $\mathbf{r} = \mathbf{r}_i$, so from (4.32) it follows:

$$G(\mathbf{r}_i,\mathbf{r}'_i) = \frac{1}{h}G_0(\mathbf{r}_i-\mathbf{r}'_i) + \\ +\frac{1}{h}\sum_\nu \varepsilon_\nu\frac{1-h}{h+\varepsilon_\nu}\frac{\varepsilon_\nu}{1+\varepsilon_\nu}\psi_\nu^{(i)}(\mathbf{r})\psi_\nu^{(i)}(\mathbf{r}') + \\ +\frac{1-h}{h}\bar{\psi}^{(i)}(\mathbf{r})\bar{\psi}^{(i)}(\mathbf{r}'_i), \tag{4.33}$$

where:

$$\bar{\psi}^{(i)}(\mathbf{r}) = \bar{\psi}^{(i)}(\mathbf{r}'_i) = \bar{\Psi}.$$

At large $\mathbf{r}_e$, from formulas (4.23) and (4.29) we find:

$$\mathbf{r}_e \to \infty: \quad G(\mathbf{r}_e,\mathbf{r}'_e) \simeq G_0(\mathbf{r}_e-\mathbf{r}'_e) \simeq -\frac{1}{4\pi r_e}; \tag{4.34}$$

$$\mathbf{r}_e \to \infty: \quad G(\mathbf{r}_e,\mathbf{r}'_i) \simeq -\bar{\psi}^{(e)}(\mathbf{r})\bar{\psi}^{(i)}(\mathbf{r}') \simeq \\ \simeq -\frac{\bar{q}}{r_e}\bar{\Psi} = -\frac{1}{4\pi r_e}. \tag{4.35}$$

In (4.35), the asymptotic behavior (2.27) for the charge function and relation (2.31) are considered.

5. BOUNDARY PROBLEMS

A number of physical problems in electrostatics, hydrodynamics of an ideal fluid, etc., leads to the following mathematical problem: outside or inside a macroscopic body of a given shape, it is necessary to find a solution to Laplace's equation that satisfies a certain condition on the surface (S) of this body. Below, two main boundary (boundary) problems will be considered: Dirichlet, where the value of the potential itself is specified on the boundary of the body, and Neumann, where the normal derivative of the potential is specified on the surface S.

To consider the formulated problems, we start from the equality:

$$\varphi(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}') =$$

$$= \nabla_{\mathbf{r}'}\{\varphi(\mathbf{r}')\nabla_{r'}G_0(\mathbf{r}-\mathbf{r}') - G_0(\mathbf{r}-\mathbf{r}')\nabla_{r'}\varphi(\mathbf{r}')\}, \quad (5.1)$$

valid inside and outside the body, if the potential $\varphi(r)$ satisfies Laplace's equation. To derive the equations necessary to solve external problems, integrate (5.1) over the volume V_e outside the body. At $\mathbf{r} = \mathbf{r}_e$ and $\mathbf{r} = \mathbf{r}_i$ respectively, we obtain:

$$\varphi^{(e)}(\mathbf{r}) = -\int k(\mathbf{r}_e, \rho')\varphi^{(e)}(\rho')d\rho' + \int G_0(\mathbf{r}_e - \rho')\chi^{(e)}(\rho')d\rho', \quad (5.2)$$

$$\int k(\mathbf{r}_i, \rho')\varphi^{(e)}(\rho')d\rho' = \int G_0(\mathbf{r}_i - \rho')\chi^{(e)}(\rho')d\rho'. \quad (5.3)$$

Here, the value of $k(\mathbf{r}, \rho')$ is defined in (3.5), and the normal derivative $\chi^{(e)}(\rho)$ in (2.6).

5.1. External Dirichlet Problem

In this problem, the potential $\varphi^{(e)}(\rho)$ itself is specified on the surface of the body. Therefore, using relation (5.3), express the value of $\chi^{(e)}(\rho)$ through the value of the potential $\varphi^{(e)}(\rho)$ on the surface S. The value of $\chi^{(e)}(\rho)$ is sought in the form:

$$\chi^{(e)}(\rho) = \sum_\nu A_{\nu D}\Phi_\nu^{(e)}(\rho) + \bar{A}_D\bar{\Phi}^{(e)}(\rho). \quad (5.4)$$

Substituting (5.4) into equation (5.3), considering (3.21) and (3.24), gives:

$$\sum_\nu A_{\nu D}\frac{\varepsilon_\nu}{1+\varepsilon_\nu}\psi_\nu^{(i)}(\mathbf{r}) + \bar{A}_D\bar{\Psi} = \int k(\mathbf{r}_i, \rho')\varphi^{(e)}(\rho')d\rho'. \quad (5.5)$$

Set here $r_i = \rho$ and replace the index v with μ:

$$\sum_\mu A_{\mu D}\frac{\varepsilon_\mu}{1+\varepsilon_\mu}\Psi_\mu(\rho) + \bar{A}_D\bar{\Psi} = \int K^{(i)}(\rho, \rho')\varphi^{(e)}(\rho')d\rho'. \quad (5.6)$$

The value of $K^{(i)}(\rho, \rho')$ is defined in (3.18). Multiply relation (5.6) successively by $\Phi_\nu^{(e)}(\rho)$ and $\bar{\Phi}^{(e)}(\rho)$, and then integrate the obtained equalities over the entire surface area of the body. Using expression (3.39) we find:

$$\int K^{(i)}(\rho, \rho')\Phi_\nu^{(e)}(\rho)d\rho = \frac{\varepsilon_\nu}{1+\varepsilon_\nu}\Phi_\nu^{(e)}(\rho'), \quad (5.7.1)$$

$$\int K^{(i)}(\rho, \rho')\bar{\Phi}^{(e)}(\rho)d\rho = \bar{\Phi}^{(e)}(\rho'), \quad (5.7.2)$$

so that:

$$A_{\nu D} = -\int \Phi_\nu^{(e)}(\rho')\varphi^{(e)}(\rho')d\rho', \quad (5.8.1)$$

$$\bar{A}_D = -\int \bar{\Phi}^{(e)}(\rho')\varphi^{(e)}(\rho')d\rho'. \quad (5.8.2)$$

Here and subsequently, the orthonormality relations (2.24), (2.31), (2.35), (2.38) are applied. As a result, we obtain:

$$\chi^{(e)}(\rho) = -\int \{\sum_\nu \Phi_\nu^{(e)}(\rho)\Phi_\nu^{(e)}(\rho') + \bar{\Phi}^{(e)}(\rho)\bar{\Phi}^{(e)}(\rho')\} \times \varphi^{(e)}(\rho')d\rho'. \quad (5.9)$$

Substituting this expression into equality (5.2) gives:

$$\varphi^{(e)}(\mathbf{r}) = -\int\{k(\mathbf{r}_e, \rho') + \sum_\nu \frac{\varepsilon_\nu}{1+\varepsilon_\nu} \times \psi_\nu^{(e)}(\mathbf{r})\Phi_\nu^{(e)}(\rho') + \bar{\psi}^{(e)}(\mathbf{r})\bar{\Phi}^{(e)}(\rho')\} \times \varphi^{(e)}(\rho')d\rho'. \quad (5.10)$$

In deriving this expression, equalities (3.20) and (3.23) were used. From relation (5.10), considering the expression (3.36) for the value of $k(r_e, \rho')$, we obtain the solution to the external Dirichlet problem in the following form:

$$\varphi_D^{(e)}(\mathbf{r}) = \int F_D^{(e)}(\mathbf{r}, \rho')\varphi^{(e)}(\rho')d\rho', \quad (5.11)$$

where:

$$F_D^{(e)}(\mathbf{r}, \rho') = -\left\{\sum_\nu \psi_\nu^{(e)}(\mathbf{r})\Phi_\nu^{(e)}(\rho') + \bar{\psi}^{(e)}(\mathbf{r})\bar{\Phi}^{(e)}(\rho')\right\}. \quad (5.12)$$

The potential $\varphi_D^{(e)}(\mathbf{r})$ outside the body obviously satisfies Laplace's equation. On the surface of the body ($\mathbf{r}_e = \rho$), due to the completeness relation (2.41), we have:

$$F_D^{(e)}(\rho, \rho') = \delta(\rho - \rho'),$$

so the equality (5.11) turns into an identity:

$$\varphi_D^{(e)}(\rho) = \varphi^{(e)}(\rho).$$

5.2. External Neumann Problem

In this case, the normal derivative of the potential $\chi^{(e)}(\rho)$ is specified on the boundary of the body S. Therefore, using equation (5.3), let us express the surface potential $\varphi^{(e)}(\rho)$ through the value of $\chi^{(e)}(\rho)$. Seek $\varphi^{(e)}(\rho)$ in the form:

$$\varphi^{(e)}(\rho) = \sum_\nu A_{\nu N}\Psi_\nu(\rho) + \bar{A}_N\bar{\Psi}(\rho). \quad (5.13)$$

Substituting (5.13) into equation (5.3), considering (3.15) and (3.8), gives:

$$\sum_\nu A_{\nu N}\frac{\varepsilon_\nu}{1+\varepsilon_\nu}\psi_\nu^{(i)}(\mathbf{r}) + \bar{A}_N\bar{\Psi} = \\ = \int G_0(\mathbf{r}_i - \rho)\chi^{(e)}(\rho)d\rho. \quad (5.14)$$

Set $\mathbf{r}_i = \rho$ here and replace the index ν with μ. The transformed equality (5.14), multiplied by $\Phi_\nu^{(e)}(\rho)$, and then by $\bar{\Phi}^{(e)}(\rho)$. Integrating the obtained relations over the entire surface area of the body, we find the decomposition coefficients (5.13):

$$A_{\nu N} = -\int \Psi_\nu(\rho')\chi^{(e)}(\rho')d\rho', \quad (5.15.1)$$

$$\bar{A}_N = -\int \bar{\Psi}(\rho')\chi^{(e)}(\rho')d\rho'. \quad (5.15.2)$$

Therefore, for $\varphi^{(e)}(\rho)$ from (5.13), we obtain:

$$\varphi^{(e)}(\rho) = \\ = -\int\{\sum_\nu \Psi_n(\rho)\Psi_\nu(\rho') + \bar{\Psi}(\rho)\bar{\Psi}(\rho')\}\chi^{(e)}(\rho')d\rho'. \quad (5.16)$$

Here:

$$\bar{\Psi}(\rho) = \bar{\Psi}(\rho') = \bar{\Psi}.$$

Substituting $\varphi^{(e)}(\rho)$ from (5.16) into (5.2) gives:

$$\varphi^{(e)}(\mathbf{r}) = \\ = -\int\left\{\sum_\nu \frac{1}{1+\varepsilon_\nu}\psi_\nu^{(e)}(\mathbf{r})\Psi_\nu(\rho') - G_0(\mathbf{r}_e - \rho')\right\}\times \\ \times \chi^{(e)}(\rho')d\rho'. \quad (5.17)$$

In deriving (5.17), relations (3.14) and (3.7) were taken into account. Using the explicit expression (3.31) for $G_0(r_e - \rho')$ in (5.17), we obtain the solution to the external Neumann problem in the following form:

$$\varphi_N^{(e)}(\mathbf{r}) = \int F_N^{(e)}(\mathbf{r},\rho')\chi^{(e)}(\rho')d\rho', \quad (5.18)$$

where:

$$F_N^{(e)}(\mathbf{r},\rho') = -\left\{\sum_\nu \psi_\nu^{(e)}(\mathbf{r})\Psi_\nu(\rho') + \bar{\psi}^{(e)}(\mathbf{r})\bar{\Psi}(\rho')\right\}.$$

The potential $\varphi_N^{(e)}(\mathbf{r})$ outside the body satisfies Laplace's equation. Calculating the normal derivative $\chi_N^{(e)}(\rho)$ of the potential (5.18), (5.19), we verify that it matches $\chi^{(e)}(\rho)$.

The goal of internal boundary (boundary) problems is to find a solution to Laplace's equation inside a certain cavity, on the surface of which the value of the potential $\varphi^{(i)}(\rho)$, itself, or its normal derivative $\chi^{(i)}(\rho)$, is specified. To derive the equations necessary for solving internal problems, integrate relation (5.1) over the volume v. As a result, at $\mathbf{r} = \mathbf{r}_i$ and $\mathbf{r} = \mathbf{r}_e$ respectively, we obtain:

$$\varphi^{(i)}(\mathbf{r}) = \int k(\mathbf{r}_i,\rho')\varphi^{(i)}(\rho')d\rho' - \\ -\int G_0(\mathbf{r}_i - \rho')\chi^{(i)}(\rho')d\rho', \quad (5.20)$$

$$\int k(\mathbf{r}_e,\rho')\varphi^{(i)}(\rho')d\rho' = \\ = \int G_0(\mathbf{r}_e - \rho')\chi^{(i)}(\rho')d\rho'. \quad (5.21)$$

5.3. Internal Dirichlet Problem

In this case, the value of the potential $\varphi^{(i)}(\rho)$ itself is specified on the surface S of the cavity. Therefore, solving equation (5.21), express the normal derivative $\chi^{(i)}(\rho)$ through the surface potential $\varphi^{(i)}(\rho)$. The value of $\chi^{(i)}(\rho)$ is sought in the form:

$$\chi^{(i)}(\rho) = \sum_\nu B_{\nu D}\Phi_\nu^{(e)}(\rho) + \bar{B}_D\bar{\Phi}^{(e)}(\rho). \quad (5.22)$$

Substituting (5.22) into (5.21) considering (3.20) and (3.23) gives:

$$\sum_\nu B_{\nu D}\frac{\varepsilon_\nu}{1+\varepsilon_\nu}\psi_\nu^{(e)}(\mathbf{r}) + \bar{B}_D\bar{\psi}^{(e)}(\mathbf{r}) = \\ = \int k(\mathbf{r}_e,\rho')\varphi^{(i)}(\rho')d\rho'. \quad (5.23)$$

Let us set here $\mathbf{r}_e = \rho$ and replace the index ν with μ, so that (5.23) takes the form:

$$\sum_\mu B_{\mu D}\frac{\varepsilon_\mu}{1+\varepsilon_\mu}\Psi_\mu(\rho) + \bar{B}_D\bar{\Psi} = \\ = \int K^{(e)}(\rho,\rho')\varphi^{(i)}(\rho')d\rho'. \quad (5.24)$$

We multiply (5.24) by $\Phi_\nu^{(e)}(\rho)$ and integrate over the entire surface area S. Then multiply (5.24) by $\bar{\Phi}^{(e)}(\rho)$ and also integrate over the surface S. As, according to (3.38),

$$\int K^{(e)}(\rho,\rho')\Phi_\nu^{(e)}(\rho)d\rho = \frac{1}{1+\varepsilon_\nu}\Phi_\nu^{(e)}(\rho'), \quad (5.25.1)$$

$$\int K^{(e)}(\rho,\rho')\bar{\Phi}^{(e)}(\rho)d\rho = 0, \quad (5.25.2)$$

then:

$$B_{\nu D} = \frac{1}{\varepsilon_\nu}\int \Phi_\nu^{(e)}(\rho')\varphi^{(i)}(\rho')d\rho', \quad \bar{B}_D = 0. \quad (5.26)$$

Therefore, from (5.22) and (5.26) it follows:

$$\chi^{(i)}(\rho) = \sum_\nu \frac{1}{\varepsilon_\nu}\Phi_\nu^{(e)}(\rho)\int \Phi_\nu^{(e)}(\rho')\varphi^{(i)}(\rho')d\rho'. \quad (5.27)$$

Substituting $\chi^{(i)}(\rho)$ from (5.27) into (5.20) gives:

$$\varphi^{(i)}(\mathbf{r}) = \int\left\{k(\mathbf{r}_i,\rho') - \sum_\nu \frac{1}{1+\varepsilon_\nu}\psi_\nu^{(i)}(\mathbf{r})\Phi_\nu^{(e)}(\rho')\right\}\times \varphi^{(i)}(\rho')d\rho'. \quad (5.28)$$

Finally, using the expression (3.37) for $k(\mathbf{r}_i,\rho')$, we obtain the solution to the internal Dirichlet problem in the following form:

$$\varphi_D^{(i)}(\mathbf{r}) = \int F_D^{(i)}(\mathbf{r},\rho')\varphi^{(i)}(\rho')d\rho', \quad (5.29)$$

where:

$$F_D^{(i)}(\mathbf{r},\rho') = -\left\{\sum_\nu \psi_\nu^{(i)}(\mathbf{r})\Phi_\nu^{(e)}(\rho') + \bar{\psi}^{(i)}(\mathbf{r})\bar{\Phi}^{(e)}(\rho')\right\}. \quad (5.30)$$

The potential $\varphi_D^{(i)}(\mathbf{r})$ satisfies Laplace's equation inside the cavity, and the boundary condition on its surface is met:

$$\varphi_D^{(i)}(\rho) = \varphi^{(i)}(\rho).$$

5.4. Internal Neumann Problem

In this problem, the normal derivative of the potential $\chi^{(i)}(\rho)$ is specified on the surface of the cavity. Therefore, in solving equation (5.21), express the surface potential $\varphi^{(i)}(\rho)$ through the value of $\chi^{(i)}(\rho)$.

The potential is sought in the form of a decomposition:

$$\varphi^{(i)}(\rho) = \sum_\nu B_{\nu N}\Psi_\nu(\rho) + \bar{B}_N\bar{\Psi}. \quad (5.31)$$

Substituting (5.31) into (5.21) considering (3.14) leads to a relationship:

$$-\sum_\nu B_{\nu N}\frac{1}{1+\varepsilon_\nu}\psi_\nu^{(e)}(\mathbf{r}) = \int G_0(\mathbf{r}_e-\rho')\chi^{(i)}(\rho')d\rho'.$$

Due to the relationship (3.7), the coefficient $\bar{B}_N$ does not enter into this expression. Set in (5.32) $\mathbf{r}_e = \rho$, and replace the index v with μ, multiply by $\Phi_\nu^{(e)}(\rho)$ and integrate over the entire surface area S of the cavity. Thus, finding the coefficient B_{vN}, for the surface potential $\varphi^{(i)}(\rho)$ from (5.31) we get:

$$\varphi^{(i)}(\rho) = \sum_\nu \varepsilon_\nu\Psi_\nu(\rho)\int \Psi_\nu(\rho')\chi^{(i)}(\rho')d\rho'. \quad (5.33)$$

Substituting $\varphi^{(i)}(\rho)$ from (5.33) into the relationship (5.20) gives:

$$\varphi^{(i)}(\mathbf{r}) = \int\left\{\sum_\nu \frac{\varepsilon_\nu^2}{1+\varepsilon_\nu}\psi_\nu^{(i)}(\mathbf{r})\Psi_\nu(\rho') - G_0(\mathbf{r}_i-\rho')\right\}\times \chi^{(i)}(\rho')d\rho'. \quad (5.34)$$

Using the expression (3.32) for $G_0(\mathbf{r}_i-\rho')$ we obtain the solution to the internal Neumann problem in the following form:

$$\varphi_N^{(i)}(\mathbf{r}) = \int F_N^{(i)}(\mathbf{r},\rho')\chi^{(i)}(\rho')d\rho', \quad (5.35)$$

where:

$$F_N^{(i)}(\mathbf{r},\rho') = \sum_\nu \varepsilon_\nu\psi_\nu^{(i)}(\mathbf{r})\Psi_\nu(\rho') + \bar{\psi}^{(i)}(\mathbf{r})\bar{\Psi}(\rho').$$

Note that integrating the Laplace equation for the potential over the volume v of the cavity gives:

$$\int_v \nabla^2\varphi^{(i)}(\mathbf{r})dr = \int \chi^{(i)}(\rho)d\rho = 0. \quad (5.37)$$

Since $\bar{\Psi}(\rho') = \bar{\Psi} = \text{const}$ is such that, due to (5.37) we have:

$$\int \bar{\Psi}(\rho')\chi^{(i)}(\rho')d\rho' = \bar{\Psi}\int \chi^{(i)}(\rho')d\rho' = 0. \quad (5.38)$$

Therefore, the product $\bar{\psi}^{(i)}(\mathbf{r})\bar{\Psi}(\rho')$ in (5.36) should be omitted.

Thus, for the potential $\varphi_N^{(i)}(\mathbf{r})$ we finally obtain:

$$\varphi_N^{(i)}(\mathbf{r}) = \int\left\{\sum_\nu \varepsilon_\nu \psi_\nu^{(i)}(\mathbf{r})\Psi_\nu(\rho')\right\}\chi^{(i)}(\rho')d\rho'. \quad (5.39)$$

The potential $\varphi_N^{(i)}(\mathbf{r})$ inside the cavity satisfies Laplace's equation. For the normal derivative of the potential $\chi_N^{(i)}(\rho)$ from (5.39), we find:

$$\chi_N^{(i)}(\rho) = -\int\left\{\sum_\nu \Phi_\nu^{(e)}(\rho)\Psi_\nu(\rho')\right\}\chi^{(i)}(\rho')d\rho', \quad (5.40)$$

where the relationship (2.10) is considered. Adding and subtracting $\bar{\Phi}^{(e)}(\rho)\bar{\Psi}(\rho')$ in the curly brackets we bring (5.40) to the form:

$$\chi_N^{(i)}(\rho) = -\int\left\{\sum_\nu \Phi_\nu^{(e)}(\rho)\Psi_\nu(\rho') + \bar{\Phi}^{(e)}(\rho)\bar{\Psi}(\rho')\right\}\chi^{(i)}(\rho')d\rho' + \bar{\Phi}^{(e)}(\rho)\int\bar{\Psi}(\rho')\chi^{(i)}(\rho')d\rho'. \quad (5.41)$$

Here, the integral in the second term according to (5.38) equals zero. Therefore, due to the completeness relation (2.41) from (5.41) it follows:

$$\chi_N^{(i)}(\rho) = \chi^{(i)}(\rho),$$

so the boundary conditions of the problem are met.

6. POLARIZABILITY TENSOR

Let us consider the following electrostatic problem: A macroscopic body with dielectric permittivity $\varepsilon^{(e)}$ is situated in a medium with dielectric permittivity $\varepsilon^{(i)}$. An external homogeneous electric field with intensity E_0 is applied to this system. In this scenario, the external (outside the body) potential $\varphi^{(e)}(\mathbf{r})$, which obeys Laplace's equation, can be represented as:

$$\varphi^{(e)}(\mathbf{r}) = -\mathbf{r}\mathbf{E}_0 + \delta\varphi^{(e)}(\mathbf{r}), \quad (6.1)$$

where $\delta\varphi^{(e)}(r)$ also obeys Laplace's equation. At large distances from the body, the "truncated" potential $\delta\varphi^{(e)}(r)$ has the following asymptotic behavior:

$$r \to \infty: \quad \delta\varphi^{(e)}(\mathbf{r}) \simeq \frac{\mathbf{p}\mathbf{r}}{r^3} + \cdots, \quad (6.2)$$

where:

$$\mathbf{p} = \widehat{\Lambda}\mathbf{E}_0 \quad (6.3)$$

is the dipole moment of the body, and $\widehat{\Lambda}$ is the polarizability tensor. The task is to find an expression for the polarizability tensor $\widehat{\Lambda}$ for a body of arbitrary shape.

This problem is solved using the method of eigenfunctions. It turns out that in this problem, the charge state does not contribute. Therefore, we will omit the charge function from the beginning. To find the potential $\varphi^{(i)}(\mathbf{r})$ inside the body, we use the results of solving the internal Dirichlet problem, according to which specifying the surface potential allows determining its $\varphi^{(i)}(\rho)$ normal derivative $\chi^{(i)}(\rho)$ according to relation (5.27), and the function $\varphi^{(i)}(\mathbf{r})$ itself according to formulas (5.29), (5.30). Therefore, setting:

$$\varphi^{(i)}(\rho) = \sum_\nu B_\nu \Psi_\nu(\rho), \quad (6.4)$$

we find:

$$\chi(i)(\rho) = \sum_\nu \frac{1}{\varepsilon_\nu}\Phi_\nu^{(e)}(\rho)\int\Phi_\nu^{(e)}(\rho')\varphi^{(i)}(\rho')d\rho' = -\sum_\nu \frac{1}{\varepsilon_\nu}B_\nu\Phi_\nu^{(e)}(\rho); \quad (6.5)$$

$$\varphi^{(i)}(\mathbf{r}) = -\sum_\nu \psi_\nu^{(i)}(\mathbf{r})\int\Phi_\nu^{(e)}(\rho')\varphi^{(i)}(\rho')d\rho' = \sum_\nu B_\nu\psi_\nu^{(i)}(\mathbf{r}). \quad (6.6)$$

For the full potential $\varphi^{(e)}(\mathbf{r})$ outside the body, the results of solving the external Dirichlet problem are unsuitable due to the divergence at $r \to \infty$ of the term $-\mathbf{r}\mathbf{E}_0$. However, they can be applied to the truncated potential $\delta\varphi^{(e)}(\mathbf{r})$. If the surface potential $\delta\varphi^{(e)}(\rho)$ is specified, its normal derivative $\delta\chi^{(e)}(\rho)$ can be determined according to equation (5.9), and the function $\delta\varphi^{(e)}(\mathbf{r})$ itself can be determined according to formulas (5.11), (5.12). Therefore, setting:

$$\delta\varphi^{(e)}(\rho) = \sum_\nu A_\nu \Psi_\nu(\rho), \quad (6.7)$$

we find:

$$\delta\chi^{(e)}(\rho) = -\sum_\nu \Phi_\nu^{(e)}(\rho)\int\Phi_\nu^{(e)}(\rho')\delta\varphi^{(e)}(\rho')d\rho' = \sum_\nu A_\nu\Phi_\nu^{(e)}(\rho); \quad (6.8)$$

$$\delta\varphi^{(e)}(\mathbf{r}) = -\sum_\nu \psi_\nu^{(e)}(\mathbf{r})\int\Phi_\nu^{(e)}(\rho')\delta\varphi^{(e)}(\rho')d\rho' = \sum_\nu A_\nu\psi_\nu^{(e)}(\mathbf{r}). \quad (6.9)$$

The full potential $\varphi^{(e)}(\mathbf{r})$ is given by formula (6.1) with $\delta\varphi^{(e)}(\mathbf{r})$ from (6.9), and for its normal derivative, we have:

$$\chi^{(e)}(\rho) = -\mathbf{n}\mathbf{E}_0 + \delta\chi^{(e)}(\rho), \tag{6.10}$$

Where n is a vector of the external normal and $\delta\chi^{(e)}(\rho)$ is determined in (6.8).

For potentials $\varphi^{(e)}(\mathbf{r})$ and $\varphi^{(i)}(\mathbf{r})$, boundary conditions (2.6) must be met, so:

$$-\rho\mathbf{E}_0 + \sum_\nu A_\nu \Psi_\nu(\rho) = \sum_\nu B_\nu \Psi_\nu(\rho), \tag{6.11}$$

$$-\mathbf{n}\mathbf{E}_0 + \sum_\nu A_\nu \Phi_\nu^{(e)}(\rho) = -h\sum_\nu \frac{1}{\varepsilon_\nu} B_\nu \Phi_\nu^{(e)}(\rho), \tag{6.12}$$

where:

$$h = \frac{\varepsilon^{(i)}}{\varepsilon^{(e)}}.$$

In (6.11), (6.12), replace the index v with μ, multiply (6.11) by $\Phi_\nu^{(e)}(\rho)$, and multiply (6.12) by $\Psi_\nu(\rho)$. Integrating the obtained equalities over the entire surface area of the body, we find:

$$\mathbf{E}_0\int \rho\Phi_\nu^{(e)}(\rho)d\rho + A_\nu = B_\nu, \tag{6.13}$$

$$\mathbf{E}_0\int \mathbf{n}\Psi_\nu(\rho)d\rho + A_\nu = -\frac{h}{\varepsilon_\nu}B_\nu. \tag{6.14}$$

Note that:

$$\int \rho\Phi_\nu^{(i)}(\rho)d\rho = \int_v \frac{\partial}{\partial x_\alpha}\left\{\mathbf{r}\frac{\partial\psi_\nu^{(i)}(\mathbf{r})}{\partial x_\alpha}\right\}dr =$$

$$= \int_v \nabla\psi_\nu^{(i)}(\mathbf{r})dr = \int \mathbf{n}\Psi_\nu(\rho)d\rho, \tag{6.15}$$

so that:

$$\int \mathbf{n}\Psi_\nu(\rho)d\rho = -\frac{1}{\varepsilon_\nu}\int \rho\Phi_\nu^{(e)}(\rho)d\rho. \tag{6.16}$$

Therefore, equality (6.14) takes the form:

$$-\mathbf{E}_0\int \rho\Phi_\nu^{(e)}(\rho)d\rho + \varepsilon_\nu A_\nu = -hB_\nu. \tag{6.17}$$

From (6.13) and (6.17), we find:

$$A_\nu = \frac{1-h}{h+\varepsilon_\nu}\mathbf{E}_0\int \rho\Phi_\nu^{(e)}(\rho)d\rho, \tag{6.18}$$

where, according to (3.29), (3.30):

$$\int \rho\Phi_\nu^{(e)}(\rho)d\rho = -4\pi\mathbf{d}_\nu\frac{\varepsilon_\nu}{1+\varepsilon_\nu}. \tag{6.19}$$

At large distances from the body, expression (6.9), taking into account the asymptotics (3.28), takes the form of (6.2), where:

$$\mathbf{p} = \sum_\nu A_\nu \mathbf{d}_\nu. \tag{6.20}$$

By substituting the coefficient A_ν from (6.18), (6.19) into formula (6.20) and comparing it with (6.3), we obtain an expression for the polarizability tensor:

$$\Lambda_{\alpha\beta} = -4\pi(1-h)\sum_\nu \gamma_\nu \frac{d_{\nu\alpha}d_{\nu\beta}}{h+\varepsilon_\nu}, \tag{6.21}$$

where:

$$\gamma_\nu = \frac{\varepsilon_\nu}{1+\varepsilon_\nu}.$$

7. CONCLUSION

The method of eigenfunctions has been presented within the framework of macroscopic electrostatics. However, eigenfunctions and eigenvalues are determined by the geometry of the problem being considered, not by its physical content. Therefore, this method can be applied in all cases where the original problem reduces to solving Laplace's equation. Such problems arise, as is well known, in the hydrodynamics of ideal fluids [12], aerodynamics [13], and in the steady-state theories of heat conduction, diffusion, conductivity, etc. For instance, in the works [10, 11, 14], the discussed method was used when considering the conductivity of a two-dimensional Rayleigh model — a thin film with a periodic distribution of circular inclusions. The application of the method made it possible to find an exact expression for the effective conductivity of the model in the area most difficult to study — the vicinity of the metal-dielectric phase transition point.

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