

POWERFUL FLARES AND MAGNETO-ELASTIC OSCILLATIONS OF MAGNETARS

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Magnetars are neutron stars with superstrong magnetic fields which can exceed 10^{15} G. Some magnetars (the so-called soft gamma-repeaters – SGRs) demonstrate occasionally very powerful processes of energy release, which result in exceptionally strong flares of electromagnetic radiation. It is believed that these flares are associated with the presence of superstrong magnetic fields. Despite many hypotheses, the mechanism of these flares remains a mystery. In afterglows of the flares, one has often observed quasi-periodic oscillations (QPOs) of magnetar emission. They are interpreted as stellar vibrations, excited by the flares, which are useful for exploring the nature of magnetar activity. The incompleteness of theories employed to interpret magnetar QPOs is discussed. Published in: *Zhurnal Experimentalnoi i Teoreticheskoi Fiziki*, Vol. 166 (7) (2024).

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1. INTRODUCTION

Pyotr Leonidovich Kapitzka, to whom this issue of *Journal of Experimental and Theoretical Physics (ZhETF)* is dedicated, made an outstanding contribution to studies of very strong magnetic fields [1]. Perhaps he would have liked magnetars – the natural laboratories of superstrong magnetic fields.

Neutron stars are the most compact of all stars. They are well known astrophysical objects, but are still fascinating because of extreme physical conditions in and around them. They contain superdense matter with superstrong magnetic fields in the presence of enormous gravitational forces. Many properties of neutron stars (for example, the equation of state and the composition of matter in inner layers) are still poorly understood.

A schematic structure of a neutron star is shown in Fig. 1. One can distinguish two main internal layers (e.g., [3]): the outer shell, often called the crust, and the inner core. At a typical neutron star mass, $M \sim 1.4 M_{\odot}$ ($M_{\odot}$ is the solar mass) its radius is $R \sim 12$ km. The crust consists mostly of ions (atomic nuclei), electrons, and (at densities $\rho \gtrsim 4 \times 10^{11} \text{ g cm}^{-3}$) free neutrons. It is ~ 1 km thick, has a mass of $\sim 0.01 M_{\odot}$. The density at the crust bottom is about half the standard nuclear density ρ_0 , with $\rho_0 \approx 2.8 \times 10^{14} \text{ g cm}^{-3}$. The atomic nuclei in the crust form usually Coulomb crystals. Beneath the crust, there is a massive and bulky core, containing liquid nuclear matter; its composition and equation of state are not reliably known. The central density of the star reaches several ρ_0 .

This paper is devoted to magnetars (as reviewed, e.g. in Ref. [4]) which are neutron stars with extraordinary strong magnetic fields. Some of them form as a special class of sources called soft gamma-ray repeaters (SGRs). Occasionally, SGRs demonstrate huge energy releases (up to $\sim 10^{46}$ erg), observed as powerful flares of electromagnetic radiation, which then fade. It is thought that these processes are driven by superstrong magnetic fields.

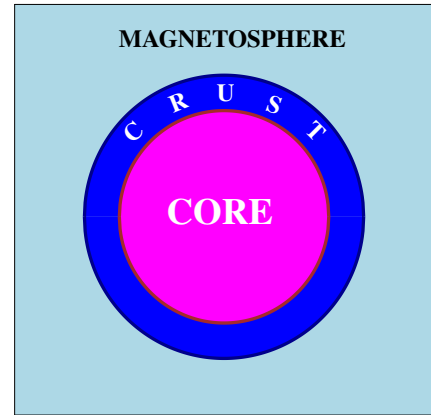


FIG. 1. Schematic structure of a neutron star. A massive and bulky core of superdense nuclear matter is surrounded by an outer shell (crust) containing an elastic crystal of atomic nuclei. A magnetar possesses superstrong magnetic fields and is surrounded by a powerful magnetosphere.

There are many models (e.g., Ref. [4]) but the nature of magnetar flares is still unknown, and it will not be discussed here.

It is important, that the flares are accompanied by observed quasi-periodic oscillations (QPOs) of the magnetar emission at certain frequencies. These are assumed to be the frequencies of stellar oscillations excited by the flares. In principle, a correct interpretation of the observed QPOs can provide useful information on the parameters of magnetars, on the strength and geometry of their magnetic fields, and on the mechanism of their flaring activity. This motivates studies of the QPO problem.

The existence of QPOs in magnetar flares was theoretically predicted by Duncan [5] in 1998. The first QPOs were discovered after observations of the giant flare of SGR 1900+14 (27 Aug. 1998) and the hyper-flare of SGR 1806–20 (27 Dec. 2004). This was done by careful pro-

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cessing observational data in the 2005–2006 [6–8], which initiated serious studies of QPOs. These observations, as well as observations of other flares of magnetars, have been processed and reprocessed many times. (e.g., [9–12]). The data on the SGR 1806–20 hyperflare seem most representative, apparently due to the exceptionally huge energy release in the event.

The observed frequencies ν of magnetar QPOs fall in a wide range from tens of Hz to several kHz. The QPOs are usually divided into low-frequency ($\nu \lesssim 150$ Hz) and high-frequency ones (at higher ν). The detection of QPOs in magnetar flares has given rise to a variety of calculations and interpretations of oscillation frequencies (e.g., [13–37] and references therein).

This paper is a logical continuation of the previous work (Refs. [20] and [37]). It provides new arguments to prove that previous calculations of magnetar QPOs have mostly dealt with an incomplete set of solutions. The paper is arranged as follows. Firstly, the formalism is briefly described in Sec. 2. Then oscillation modes and torsional oscillations of non-magnetic are outlined (in Secs. 3 and 4, respectively). In Sec. 5 we discuss magneto-elastic oscillations assuming that magnetic field effect is sufficiently weak. In Sec. 6 this case is studied for a pure dipole magnetic field in the stellar crust, and the consideration is extrapolated to higher magnetic fields. In Sec. 7, the possibility of applying the results for interpreting the observed QPOs is discussed. In Sec. 8 the results are summarized and unsolved problems are formulated.

2. FORMALISM

Oscillations of magnetars are described by the standard formalism of magneto-elastic oscillations of neutron stars. The formalism is well known (e.g., [26]); it is sufficient to outline the basic points. For simplicity, the equations are presented neglecting relativistic effects. These effects will be included in Sec. 5. Magneto-elastic oscillations are mediated by elastic forces of crystal lattice in the stellar crust and by elastic deformations (Alfvén perturbations) of magnetic field lines everywhere where the field is present.

The star is assumed to have a stationary magnetar field $\mathbf{B}(\mathbf{r})$ ($\sim 10^{14} - 10^{16}$ G), which is not strong enough to cause a noticeable distortion of stellar shape from the spherical one. The oscillation equations are obtained by linearizing the equations of motion of magnetized matter assuming the field is frozen into the matter. The unperturbed configuration of the star is thought to be spherically symmetric. Under these conditions, it is sufficient to study the oscillations of incompressible matter where matter elements move only along spherical surfaces. Then the perturbations of pressure and density are absent, and the emission of gravitational waves is suppressed. The perturbations excite small velocities of matter elements $\mathbf{v}(\mathbf{r}, t)$, small displacements of these elements $\mathbf{u}(\mathbf{r}, t)$, and small variations of magnetic magnetic

field $\mathbf{B}_1(\mathbf{r}, t)$. All these variations oscillate in time as $\exp(i\omega t)$, where $\omega = 2\pi\nu$ is the angular oscillation frequency, and ν is the cyclic frequency. This overall oscillating factor in the equations can be dropped, leading to the stationary wave equation for small (complex) amplitudes $\mathbf{u}(\mathbf{r})$ and $\mathbf{B}_1(\mathbf{r})$, and for the oscillation frequency ω :

$$\rho\omega^2\mathbf{u} = \mathbf{T}_\mu + \mathbf{T}_B. \quad (1)$$

Here $\mathbf{T}_\mu$ and $\mathbf{T}_B$ are the volume densities of forces (with minus sign) determined, respectively, by the crystal elasticity and magnetic field stresses. In the first case

$$\mathbf{T}_{\mu i} = -\frac{\partial\sigma_{ik}}{\partial x_k}, \quad \sigma_{ik} = \mu \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right), \quad (2)$$

where σ_{ik} is the tensor of shear deformations and μ is the shear viscosity (in the isotropic crystal approximation). In the case of magnetic forces one has

$$\mathbf{T}_B = \frac{1}{4\pi} \mathbf{B} \times \text{curl} \mathbf{B}_1, \quad \mathbf{B}_1 = \text{curl}(\mathbf{u} \times \mathbf{B}). \quad (3)$$

These equations should be supplemented with boundary conditions. Since the crystal exists only in the stellar crust, the radial components of viscous stresses must vanish at the outer and inner boundaries of the crust. The conditions for the magnetic field depend on the formulation of the problem. Alfvén perturbations can propagate into the core and magnetosphere of the star.

3. GENERAL REMARKS

Let us begin with a few remarks. It is well known that the shear modulus μ determines characteristic propagation speed v_μ of elastic shear deformations in a crystal. As follows from calculations (e.g., [38]), these deformations are mainly located near the bottom of the crust, at $\rho \sim 10^{14}$ g cm $^{-3}$. Then, under typical conditions, one comes to the estimate

$$v_\mu \sim \sqrt{\mu/\rho} \sim 10^8 \text{ cm s}^{-1}. \quad (4)$$

As for magnetic perturbations, they propagate with the Alfvén velocity v_A which, for the same conditions, can be estimated as

$$v_A = \frac{B}{\sqrt{4\pi\rho}} \sim 3 \times 10^7 B_{15} \text{ cm s}^{-1}, \quad (5)$$

where B_{15} is the magnetic field in units of 10^{15} G. The velocity v_A can noticeably decrease within the star and increase toward the surface.

The velocities (4) and (5) become close at

$$B \sim B_\mu \sim 3 \times 10^{15} \text{ G}. \quad (6)$$

This characteristic field strength reveals the existence of three regimes of magneto-elastic oscillations (Table I).

TABLE I. Three regimes of magneto-elastic oscillations of magnetars.

Regime	Condition	Leading mechanism
I	$B \ll B_\mu$	shear waves in crystalline matter
II	$B \sim B_\mu$	shear and Alfvén waves
III	$B \gg B_\mu$	Alfvén waves

In regime I, the oscillations are mainly regulated by elastic shear waves in the stellar crust; Alfvén perturbations are driven by these elastic shear stresses and weakly affect the oscillation frequencies. Such oscillations are almost completely localized in the crust being determined by the microphysics of matter and by the properties of $\mathbf{B}(\mathbf{r})$ in the crust.

The efficiencies of shear and Alfvén waves in regime II are comparable. Alfvén perturbations can propagate beyond the crust (e.g. [13, 14]) and spread over the whole star. Their calculation requires knowledge of the entire microphysics of the magnetar, which contains many uncertainties, including the equation of state, superfluidity and superconductivity of the stellar core, as well as magnetic field configuration within it.

Finally, in regime III, the oscillations are mainly regulated by Alfvén waves. The elasticity of crystal becomes nearly or even fully negligible (e.g., [17, 19]).

This classification of oscillations is schematic. In particular, it does not take into account possible forbidden frequency intervals of Alfvén oscillations in the stellar core (e.g., [26]) in which case the oscillations can be locked in the crust even at very strong magnetic fields. The effects of penetration of Alfvén perturbations from the crust to the core and back can also be important. They can lead to frequency variations, damping, and loss of coherence of crustal oscillations.

The employed approximation of incompressible magneto-elastic oscillations (Sec. 2) also deserves a comment. Types of neutron star oscillations are numerous (e.g., [41] and references therein). Magneto-elastic oscillations are suitable because their frequencies are low enough to explain magnetar QPOs. The shear and Alfvén velocities, v_μ and v_A , are generally much lower than the speed of ordinary sound, that is determined by the full pressure of dense stellar matter. Frequencies of oscillations of many types are higher than magneto-elastic ones.

4. ELASTIC OSCILLATIONS OF NON-MAGNETIC CRUST

Such oscillations are often called torsional. They are basic for studying magneto-elastic oscillations. Their theory began in the 1980s in the classical works of Hansen and Cioffi [39], Schumaker and Thorne [40], and McDermott et al. [41] long before the discovery of magnetar QPOs. After the discovery, the interest to the theory was renewed (e.g., [27, 29, 36, 42–48] and references therein).

Any torsion oscillation mode is characterized by three quantum numbers: (1) $n = 0, 1, 2, \dots$ is the number of radial nodes of the wave function, (2) the orbital number ℓ , which in this problem runs the values $\ell = 2, 3, \dots$, (3) the azimuthal number m which takes integer values from $-\ell$ to ℓ .

In the spherical coordinates (r, θ, ϕ) , a stationary wave function $\mathbf{u}(\mathbf{r})$ has only two non-trivial components: $\mathbf{u}_\phi$ and $\mathbf{u}_\theta$ (since $\mathbf{u}_r = 0$). These can be written as (e.g., [37])

$$u_\phi(r, \theta, \phi) = rY(r) e^{im\phi} \frac{dP_\ell^m}{d\theta}, \quad (7)$$

$$u_\theta(r, \theta, \phi) = rY(r) e^{im\phi} \frac{imP_\ell^m}{\sin\theta}, \quad (8)$$

where $P_\ell^m(\cos\theta)$ is an associated Legendre polynomial, and $Y(r) = Y_{n\ell}(r)$ is a dimensionless radial wave-function satisfying the equation

$$Y'' + \left(\frac{4}{r} + \frac{\mu'}{\mu}\right) Y' + \left[\frac{\rho}{\mu} \omega^2 - \frac{(\ell+2)(\ell-1)}{r^2}\right] Y = 0. \quad (9)$$

Prime means derivative with respect to r . These oscillations are localized in the crystalline crust, $r_1 \leq r \leq r_2$, where r_1 is the radius of the crust-core interface, and r_2 is the outer radius of the crystallization zone which is very close to the radius of the star. At both boundaries, radial elastic stresses should vanish, $Y'(r_1) = Y'(r_2) = 0$. The frequencies of torsional oscillations are degenerate in m : $\omega = \omega_{\mu n \ell}$ (the index μ indicates the elastic shear nature of these oscillations); the functions $Y(r)$ are independent of m . The value $Y_0 = Y(r_2)$ characterises the angular amplitude of oscillations (in radians) at the outer edge of the crystallization region. If $m = 0$, crustal matter oscillates only along parallels ($\mathbf{u}_\theta = 0$), but at $m \neq 0$ there appear meridional motions. The value of m strongly influences the geometry of displacements $\mathbf{u}(\mathbf{r})$ and the angular dependence of the energy density of oscillations. A specific stellar model affects only $Y(r)$, while angular dependences of $\mathbf{u}(\mathbf{r})$ stay standard.

Torsional oscillations of neutron stars are divided into fundamental ($n = 0$) and ordinary ($n > 0$) ones. For the fundamental oscillations, a very good approximation is the weak deformability of the crystal, in which case Y is almost independent of r (e.g., [38]). In this case, $\omega_{\mu 0 \ell} \approx \frac{1}{2} \omega_{\mu 0} \sqrt{(\ell+2)(\ell-1)}$, where $\omega_{\mu 0}$ is the basic frequency (at $\ell = 2$), the lowest for all torsion oscillations.

The frequencies of ordinary torsion oscillations ($n > 0$) are higher and strongly increase with increasing n . At a fixed n , there is a bunch of close frequencies which grow weakly with increasing ℓ (showing "fine splitting" with respect to ℓ). Corresponding wave functions $Y_{n\ell}(r)$ depend on ℓ rather weakly (e.g., [49]). Since torsional oscillation frequencies do not depend on m , one usually sets $m = 0$ for finding the oscillation spectrum, without using the states with $m \neq 0$.

Torsional oscillations may carry a lot of energy. For example, let us choose a neutron star model with a nucleonic core and the modern BSk21 equation of state of

dense matter (described, e.g., in Ref. [50]). For a $1.4 M_\odot$ neutron star, the stellar radius is $R = 12.6$ km and the crust-core radius is $r_1 = 11.55$ km. According to the results of Ref. [49], the oscillation energy of the basic mode ($n = 0$, $\ell = 2$, $\nu_{\mu 0} = 23.0$ Hz) is $E_{\text{vib}} \approx 10^{49} Y_0^2$ erg. At a swing angle $\approx 0.1^\circ$ ($Y_0 \sim 1.7 \times 10^{-3}$ rad) of oscillations at the outer edge of the crystalline crust, we get $E_{\text{vib}} \sim 3 \times 10^{43}$ erg. In this case, shear stresses in a vibrating crust are still far from the crystal-breaking limit [38].

5. OSCILLATIONS DOMINATED BY ELASTICITY OF THE CRUST

This is the simplest regime I of magneto-elastic oscillations (Table I). In this case magnetic fields are not too high ($B \ll B_\mu$) and can be taken into account by perturbation theory, considering the wave functions of pure torsional oscillations (Sec. 4) as zero-order wave functions, and the quantity $\mathbf{T}_B$ in Eq. (1) as a small perturbation. In numerous studies of magneto-elastic oscillations (e.g., [13–16, 18, 21–26, 28, 30, 31, 33, 35]), the states with $m \neq 0$ have been ignored. In this way one dealt with incomplete spectrum of magneto-elastic oscillations.

The exceptions were the paper by Shaisultanov and Eichler [20] and the recent paper [37]. The authors of Ref. [20] argued that the magnetic field removes the degeneracy of torsion frequencies. In a magnetic field, these frequencies should split into a series of frequencies, which can be treated as the Zeeman effect in magnetars. The effect was correctly described and evaluated, but the work did not attract much attention. Ref. [37] was devoted to developing these ideas. It proposed a simple algorithm for calculating the oscillation frequencies in the first-order perturbation theory for a wide class of $\mathbf{B}$ -fields in the magnetar crust. For illustration, the Zeeman splitting of fundamental oscillations ($n = 0$) in a dipole crustal magnetic field at $2 \leq \ell \leq 5$ was calculated.

Here we extend the consideration of magneto-elastic oscillations in the first-order perturbation theory for the fundamental modes ($n = 0$). The details of the theory were presented in [37]. Here we mention them only briefly. In the formulated approach, it is sufficient to assume that the oscillations are localized in the elastic crust. As in Ref. [37], we assume that the crustal magnetic field is axially symmetric about the magnetic axis: only the field components $B_r(r, \theta)$ and $B_\theta(r, \theta)$ are different from zero. In this case

$$\omega_{\ell m}^2 = \omega_{\mu \ell}^2 + \omega_{B \ell m}^2, \quad (10)$$

where $\omega_{\mu \ell}$ is the frequency of purely torsional oscillations (Sec. 4), and $\omega_{B \ell m}$ is the small ‘magnetic’ correction; ℓ and m have the same meaning as in the wave functions of zero-order approximation; see Eqs. (7) and (8).

The expressions for $\omega_{\mu \ell}$ and $\omega_{B \ell m}$ are given in [37]. In Sec. 6 of Ref. [49] it is also described which changes

should be introduced to the theory to account for relativistic effects. According to [37],

$$\omega_{\mu \ell}^2 = \frac{(1 - x_{g*}) \int_{\text{crust}} dV \mu}{\int_{\text{crust}} dV (\rho + P/c^2) r^2}, \quad (11)$$

$$\omega_{B \ell m}^2 = \frac{(1 - x_{g*}) \frac{1}{4\pi} \int_{\text{crust}} dV I_B}{\Xi(\ell, m) \int_{\text{crust}} dV (\rho + P/c^2) r^2}. \quad (12)$$

Here P is the pressure of dense matter; c is the speed of light; $dV = r^2 dr \sin \theta d\theta d\phi$ is the volume element in the approximation of locally flat crust; integration is over crystalline matter. The factor $(1 - x_{g*})$ approximates the gravitational redshift of a squared oscillation frequency for a distant observer, $x_{g*} = 2GM_*/(c^2 r_*)$, G is the gravitational constant. Furthermore, r_* is the radius of any point in the crust (results being almost independent of its particular choice [49]), M_* is the gravitational mass inside a sphere of radius r_* . The quantity

$$\Xi(\ell, m) = \frac{2\ell(\ell+1)(\ell+m)!}{(2\ell+1)(\ell-m)!} \quad (13)$$

is a convenient normalization factor, and I_B is a combination of B_r , B_θ , P_ℓ^m and their derivatives (see Eq. (18) in [37]) quadratic in the magnetic field, making $\omega_{B \ell m}^2 \propto B^2$. In this case, the oscillations are located in the crust and do not depend on the $\mathbf{B}(\mathbf{r})$ configuration outside the crust.

6. CASE OF THE DIPOLE FIELD

By way of illustration, following Ref. [37], we consider a purely dipole magnetic field in the stellar crust. Then $B_r = B_0 \cos \theta (R/r)^3$ and $B_\theta = \frac{1}{2} B_0 \sin \theta (R/r)^3$. Here B_0 is the field strength at the magnetic pole on the stellar surface. The field removes degeneracy of frequencies $\omega_{\mu \ell}$, but only partly: according to (10) the frequency $\omega_{\mu \ell}$ splits into a series of $\ell + 1$ components $\omega_{\ell m}$, where $m = 0, 1, \dots, \ell$. The frequency $\omega_{\ell 0}$ appears non-degenerate, while the frequencies with $m > 0$ remain degenerate twice (correspond to $\pm m$ states). The Zeeman splitting is determined by $\omega_{B \ell m}$ given by Eq. (12). For the pure dipole field

$$I_B = -\frac{B_0^2}{4} \left(\frac{R}{r}\right)^6 \left[P'^2 (1 + 3 \cot^2 \theta) - 3P' P'' \cot \theta + P' \times P''' + \frac{m^2}{\sin^2 \theta} (-P'^2 - 10 P P' \cot \theta + 8 P^2 \cot^2 \theta) \right]. \quad (14)$$

Here $P = P_\ell^m(\cos \theta)$, prime denotes differentiation with respect to θ . Then

$$\omega_{B \ell m}^2 = \frac{B_0^2 r_2^3 [(r_2/r_1)^3 - 1]}{12\pi \int_{r_1}^{r_2} dr r^4 (\rho + P/c^2)} \zeta_{\ell m}, \quad (15)$$

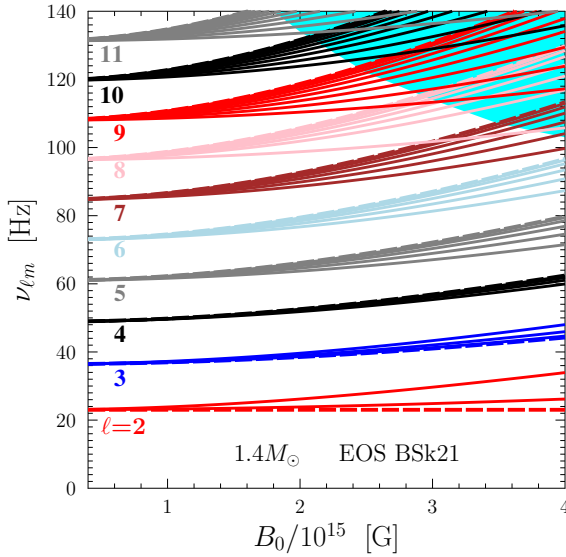


FIG. 2. The frequencies of magneto-elastic oscillations of a $1.4 M_\odot$ neutron star versus the field strength B_0 at the magnetic pole on the stellar surface. Each series of frequencies corresponds to a fixed $\ell = 2, \dots, 11$ and contains a bunch of $\ell + 1$ Zeeman components. The components with $m = 0$ are shown by dashed lines. The region which contains many quasi-crossings of Zeeman components is darkened (as detailed in the text).

where

$$\zeta_{\ell m} = \frac{1}{B_0^2 \Xi(\ell, m)} \int_0^\pi \sin \theta d\theta I_B(R, \theta). \quad (16)$$

Our Eqs. (14) and (15) correspond to Eqs. (26) and (27) in Ref. [37]. The latter contain typos, which are corrected here. All calculations in Ref. [37] were performed using correct formulae.

In Ref. [37] the factors $\zeta_{\ell m}$ were calculated and approximated by

$$\zeta_{\ell m} = c_0(\ell) + c_2(\ell)m^2 \quad (17)$$

at $\ell \leq 5$; the values of $c_0(\ell)$ and $c_2(\ell)$ were tabulated. Now the values of $\zeta_{\ell m}$ have been calculated up to $\ell = 15$. Corresponding values of $c_0(\ell)$ and $c_2(\ell)$ can be fitted as

$$c_0(\ell) = 0.721 [(\ell - 2)(\ell + 1)]^{0.954}, \quad (18)$$

$$c_2(\ell) = \frac{2}{3} - \frac{0.766 (\ell - 2)^{1.09}}{1 + 0.532 (\ell - 2)^{1.15}}. \quad (19)$$

The fit accuracy is a few per cent, which seems quite satisfactory. Equations (17)–(19) are valid for a dipole magnetic field in the crust; fields of other configurations will be studied separately.

The results for dipole fields are illustrated below, extending thus the consideration of Ref. [37] to a wider frequency range.

Figure 2 shows the dependence of oscillation frequencies on B_0 at $n = 0$, $\ell = 2, \dots, 11$ and different m . As in Ref. [37], we consider the $1.4 M_\odot$ star with the BSk21 equation of state, mentioned in Sec. 4. Calculations are performed using Eqs. (10), (11) and (15). Figure 2 is similar to Fig. 1b from Ref. [37], but covers wider frequency range $\nu \leq 140$ Hz (instead of 80 Hz in [37]).

According to the results of Sec. 3, Fig. 2 shows two regimes of magneto-elastic oscillations: regime I of field strengths much smaller than $B_\mu \sim 3 \times 10^{15}$ G, and regime II of intermediate field strengths (Table I). The equations employed are strictly valid only in regime I. In the figure, they are extrapolated to the intermediate regime. The possibility of such extrapolation requires confirmation (see below).

At $B_0 \leq 4 \times 10^{14}$ G and $\nu < 140$ Hz, Fig. 2 shows 10 frequencies of fundamental torsional oscillations (Sec. 4) which are actually unaffected by the magnetic field. However, as B_0 grows up, each of these frequencies splits noticeably into Zeeman components: 10 initial frequencies decompose into 75 branches.

At not too high B_0 , one can clearly see 10 separate bunches of curves corresponding to certain ℓ . The oscillation frequencies in each bunch differ by the values of m . In agreement with the results of [37], the branches of oscillations with $m = 0$ (dashed lines) at $\ell = 2$ and 3 lie below other branches in a bunch, while at higher ℓ they become higher than the others (this inversion is possibly specific for the dipole magnetic field). The higher ℓ , the richer the splitting, and the smaller the value of B_0 at which this splitting starts to be visible.

At $\ell > 3$, the lowest branch of oscillations in any bunch corresponds to the highest $m = \ell + 1$. Interestingly, as ℓ increases, such curves become more horizontal and depend weaker on B_0 . In other words, at high m the frequencies $\nu_{\ell m}(B_0)$ approach the frequencies of torsional oscillations $\nu_{\mu \ell}$ of a non-magnetic star (Sec. 4).

Starting from $B_0 \gtrsim 1.5 \times 10^{15}$ G and $\ell \sim 11$, in the upper right corner of Fig. 2, there appears a special region of frequencies and magnetic fields in which the magneto-elastic oscillations behave in a complicated way. If B_0 increases to the highest depicted values (4×10^{15} G), this region descends to $\nu \sim 90$ Hz (and at higher B_0 it will descend further). In this region the two effects, neglected in the calculations, can be especially important.

Firstly, with increasing B_0 , the oscillation modes from different bunches begin to show quasi-crossings (Fig. 2). The identity of individual bunch is lost, and the region becomes densely filled with allowed oscillation frequencies. The behavior of curves near quasi-crossing points requires further analysis. As usual, in the vicinity of these points, the oscillations of converging modes interact with each other, and their frequencies are distorted.

Secondly, in the presence of pronounced damping and loss of coherence of crustal oscillations due to the transfer of vibrational energy by Alfvén waves into the stellar core, the oscillation modes can acquire finite shifts and widths (this effect is expected to be amplified with in-

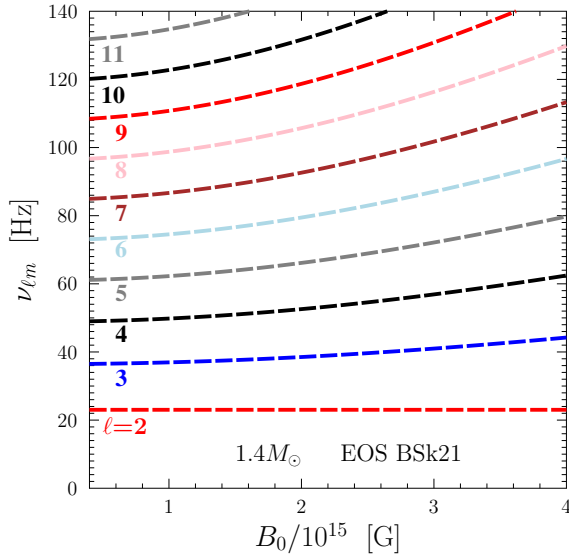


FIG. 3. Same as in Fig. 2, but we left only the oscillations with $m = 0$, which were considered in the majority of publications.

creasing B_0 and ν). The frequencies of crustal oscillations are thus capable of blurring and shifting. However, there can exist forbidden frequency intervals (e.g., [26] and references therein), which may prevent penetration of Alfvénic waves into the core.

It is clear that both effects are mutually related and require self-consistent consideration. It is impossible to correctly calculate quasi-crossings without a reliable theory of interaction between crustal oscillations and Alfvén perturbations in the core. Great efforts have been spent on the construction of such a theory (e.g., [13–15, 17, 21–26, 28, 30, 31, 33, 35]) but only for axially symmetric perturbations ($m = 0$). There is no theory at $m \neq 0$, it is a difficult task for future studies. It seems that both effects are most important in the special region of high frequencies and fields, while at lower ν and B_0 they are weaker.

7. DISCUSSION

All the above effects can be important for interpreting observed frequencies of magnetar QPOs. Below we extend the interpretation of QPOs with complete theoretical spectrum (it was started in Ref. [37]) using the data from the hyperflare of SGR 1806–20 and the giant flare of SGR 1900+14. Now we add a few higher-frequency QPOs. The unsolved problems of quasi-crossing of magneto-elastic oscillation frequencies and the interaction of crustal oscillations with Alfvén oscillations in the stellar core are not considered here. Therefore, as in Ref. [37], our consideration is illustrative and may be particularly inaccurate at sufficiently high ν and B_0 .

The available observational data on magnetar QPOs have been analyzed many times. The results are summarized, for example, in Ref. [35]. They have been widely used by many authors and will be used below. An exception is Ref. [12], whose authors expressed doubt in significance of measured low-frequency QPOs based on Bayesian model-independent extraction of noise; their conclusion requires further confirmation.

It has already been noted that in many interpretations the oscillation modes with $m \neq 0$ were ignored. For illustration, Fig. 3 presents only the frequencies of $m = 0$ oscillation modes, instead of all of them in Fig. 2. Of the 75 modes shown in Fig. 2, only 10 are left. Clearly, they constitute an incomplete set of theoretical curves. Using such a set, an interpretation of observations can be questionable. In particular, all quasi-crossings of oscillation modes in Fig. 3 disappear.

The reason for neglecting solutions with $m \neq 0$ is that the displacements $\mathbf{u}(\mathbf{r})$ of oscillating stellar matter for such solutions depend not only on r and θ , but also on the angle ϕ ; see, e.g., Eqs. (7) and (8). In other words, for axially symmetric magnetic fields $\mathbf{B}(r, \theta)$, considered by most researchers, the perturbed quantities $\mathbf{u}$ and $\mathbf{B}_1$ at $m \neq 0$ turn out to be axially asymmetric. However, the axial symmetry of perturbations was usually postulated leading to the loss of solutions with $m \neq 0$.

Let us add that, as seen from Fig. 3, the frequency ν_{20} (the lowest dashed line) does not depend on B at all (see also Ref. [37]). This result is valid in the first-order perturbation theory for weak dipole crustal magnetic fields (Sec. 5). In reality, it means that an expansion of $\nu_{20}(B_0)$ in powers of B_0^2 should not contain the B_0^2 -term. Higher-order terms can be present, although their calculation requires much effort. But according to Ref. [16], devoted to oscillations of a neutron star crust with a dipole field, the B_0^2 -term is not zero. This paradox was resolved by noting [26], that the solution in [16] had been sought by expanding u_ϕ into a series of functions (7) with different ℓ at $m = 0$. The sum over ℓ in [16] was artificially truncated, that was actually equivalent to solving an exact non-dipole magnetic field problem. This does not prove the B_0^2 -term is nonzero.

The largest number of QPOs was detected by processing observations of the hyper-flare of SGR 1806–20. The low-frequency QPOs, which are discussed below, were detected at 18, 26, 30, and 150 Hz, and (with lower confidence) at 17, 21, 36, and 59 Hz.

Can these QPOs be interpreted as fundamental magneto-elastic oscillations of a single star (with the same mass, radius, and internal structure) possessing the same crustal dipole field? This question was raised in Ref. [37], where theoretical calculations were limited to $\ell \leq 5$ and could explain QPO frequencies $\nu \leq 60$ Hz. It turned out that for a $M = 1.4 M_\odot$ star, only three of seven such frequencies (17, 18, 21, 26, 30, 36, and 59 Hz) could be interpreted: 26, 30, and 59 Hz, assuming $B_0 \approx (3.2 - 3.4) \times 10^{15}$ G (Fig. 2a in [37]). The lowest-frequency QPOs could not be explained in this

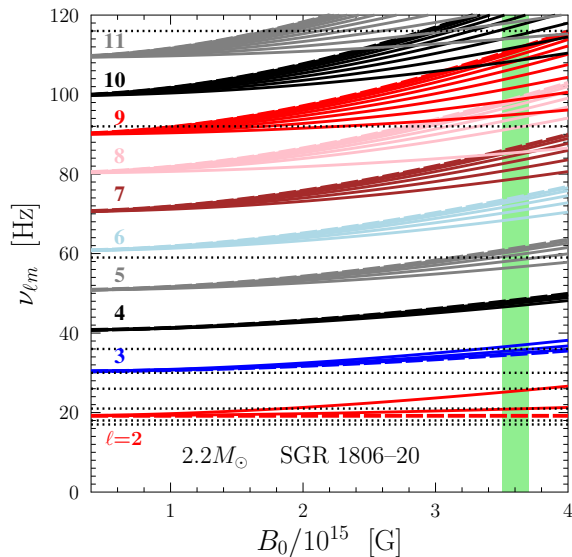


FIG. 4. Same as in Fig. 2, but for a $2.2 M_{\odot}$ star compared to QPO frequencies (dotted horizontal lines) observed from the hyper-flare of SGR 1806–20. The vertical green band shows a possible range of B_0 , simultaneously consistent with some observed QPOs (as detailed in the text).

way. However, leaving the same equation of state of the neutron star matter (BSk21), but increasing the stellar mass to $2.2 M_{\odot}$ (with the $2.27 M_{\odot}$ maximum mass limit), one could slightly lower all theoretical frequencies due to a stronger gravitational redshift of oscillation frequencies for a more massive (and compact) star. In this case (Fig. 2b in [37]), it was possible to explain six frequencies except for one (30 Hz), assuming $B_0 \approx (3.5 - 3.7) \times 10^{15}$ G.

By adding new calculations of this paper (up to $\ell = 11$), it is possible to explain all but one (30 Hz) of the observed low-frequency QPOs of the SGR 1806–20 hyperflare assuming the same field $B_0 \approx (3.5 - 3.7) \times 10^{15}$ G as in the [37]. This is shown in Fig. 4, that is similar to Fig. 2b in [37] but extended now to 120 Hz. An explanation for the highest selected QPO frequency (150 Hz) is not shown to simplify the figure, but it is evident because of very densely spaced theoretical oscillation branches at $\nu > 90$ Hz. We do not worry on the failure to explain the 30 Hz QPO [37]: no attempt has been made to seriously explain the observations. The $M = 2.2 M_{\odot}$ model was chosen as an example and was not varied. The required magnetic field B_0 corresponds to regime II (Sec. 3), where quantitative accuracy of theoretical frequencies can be questioned. In addition, a purely dipole magnetic field has been assumed, whereas possible deviations from pure dipole can change theoretical results. In any case, the proposed complete theoretical set of frequencies of magneto-elastic oscillations greatly simplifies theoretical interpretation of low-frequency magnetar QPOs.

Moreover, in Ref. [37] we tried to interpret the QPOs observed in the giant flare of SGR 1900+14. Four low-

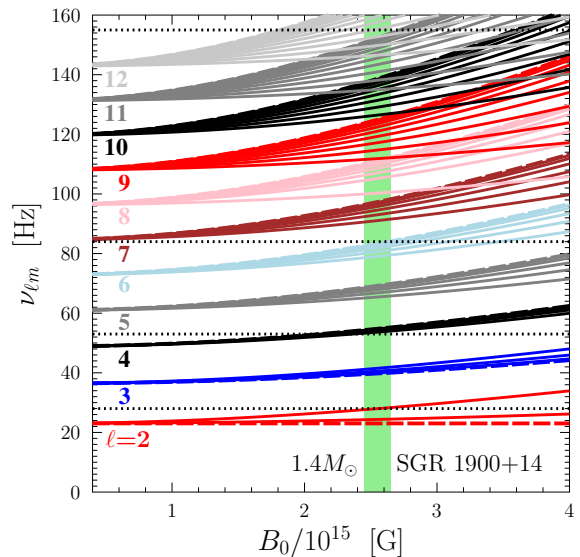


FIG. 5. Same as in Figs. 2 and 4, compared with the QPO frequencies (dotted horizontal lines) observed from the giant flare of SGR 1900+14.

frequency QPOs (28, 53, 84, and 155 Hz) were detected. The two lowest frequencies were easily explained by the $1.4 M_{\odot}$ neutron star model (Fig. 3 in [37]). By taking the same model and now increasing the theoretical frequencies to 160 Hz, it is possible to explain (Fig. 5) all four QPOs with $B_0 \approx (2.42 - 2.62) \times 10^{15}$ G. Just as for Fig. 4, a more serious interpretation seems premature.

Let us emphasize that $B \sim B_{\mu}$ has often been treated as a valid estimate of magnetar fields for various reasons (e.g., [35]).

8. CONCLUSIONS

We have attempted to develop the theory of magneto-elastic oscillations of magnetars. We have employed the standard assumption that these oscillations are excited during flares of magnetars and are observed as quasi-periodic oscillations (QPOs) at the decay phases of the flares (Sec. 1 and references therein). Correct interpretation of observations can provide useful information about the parameters of magnetars, their magnetic fields, and the nature of their flares.

Following the results of Refs. [20, 37], the completeness of theoretical QPO models has been studied, especially because many previous works neglected Zeeman splitting of magnetar oscillations. We have used a simple model of low-frequency magneto-elastic oscillations without nodes of radial wave function in the magnetar crust assuming a purely dipole crustal magnetic field. We have shown that neglecting the Zeeman effect leads to essentially incomplete set of oscillation modes. In the case of axi-

ally symmetric $\mathbf{B}(\mathbf{r})$, this simplification reproduces only the oscillations accompanied by axially symmetric vibrational displacements of matter elements $\mathbf{u}(\mathbf{r})$ and magnetic field $\mathbf{B}_1(\mathbf{r})$. In this way it misses a wide range of oscillations modes in which the displacements $\mathbf{u}(\mathbf{r})$ and $\mathbf{B}_1(\mathbf{r})$ are axially asymmetric. We have demonstrated that the full set of oscillations gives a qualitatively different oscillation spectrum and can significantly change theoretical interpretation of observed QPOs.

Therefore, the construction of complete set of magneto-elastic oscillations has only begun. Serious efforts are needed to complete it. Here are some of the problems.

Even in the most reliable regime I of relatively low magnetic fields, only low-frequency oscillations have been considered, without nodes of wave function along the radius. Generalization to the case of oscillations with nodes ($n > 0$) can be done without difficulty. Instead of the dipole field, it is easy to study poloidal magnetic fields of other types. It is also easy to consider the case in which a toroidal magnetic field is also present. In addition, our results are obtained in the approximation of a locally flat stellar crust (e.g., [37, 49]). It would be useful to solve the problem in full General Relativity. This would be especially important for the cases in which Alfvén perturbations propagate outside the crust.

The oscillation regime I, which has been studied rather reliably, is insufficient for interpreting the observations. It seems that the intermediate regime II ($B \sim B_\mu$) is more important for this purpose. Such oscillations can penetrate into the core of the star, which makes the above consideration quantitatively inaccurate (although it may be qualitatively applicable, especially for lowest frequencies). A firm study of oscillations in this regime is more complicated because the calculations should include microphysics of the stellar core for many possible models (superfluidity and superconductivity in the core, different magnetic field configurations there, etc.). There is also an important problem of interaction of Alfvén oscil-

lations in the core with crustal oscillations. The energy of crustal vibrations can flow into the core, which can lead to damping and loss of coherence of the crustal oscillations. All these effects have been studied for oscillations induced by axially symmetric perturbations. The important case of axially asymmetric perturbations has not been explored.

Another area of research is to improve the microphysics of neutron star matter that affects magnetar oscillations. In particular, one can improve calculations of the shear modulus in the crust, consideration of the delicate effects of superfluidity and superconductivity of crustal matter, nuclear interactions, nuclear pasta effects at the bottom of the crust, etc. (see, e.g., [27, 29, 36, 37, 43–49, 51]).

Finally, it is appropriate to list the main original results of this work. The studies of low-frequency magneto-elastic oscillations in Ref. [37] are extended to higher frequencies (Sec. 6, Fig. 2). Simple approximations are derived for the coefficients $c_0(\ell)$ and $c_2(\ell)$, Eqs. (18) and (19). They allow one to calculate the oscillation spectrum at $\nu \lesssim 150$ Hz. Quasi-crossings of oscillation frequencies with different ℓ are pointed out in the darkened region in Fig. 2, where allowable frequencies are densely spaced, but in fact can quickly decay through interaction of crustal and Alfvén oscillations of the core. Following Ref. [37], possible interpretations of low-frequency QPOs, observed in the hyperflare of SGR 1806–20 (Fig. 4) and the giant flare of SGR 1900+14 (Fig. 5), are further discussed. The necessity for joint consideration of quasi-crossings of modes and interactions of crustal oscillations with Alfvén perturbations in the stellar core is stressed. It is stated that, despite considerable efforts of many authors, the theory of magneto-elastic oscillations is far from being completed.

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