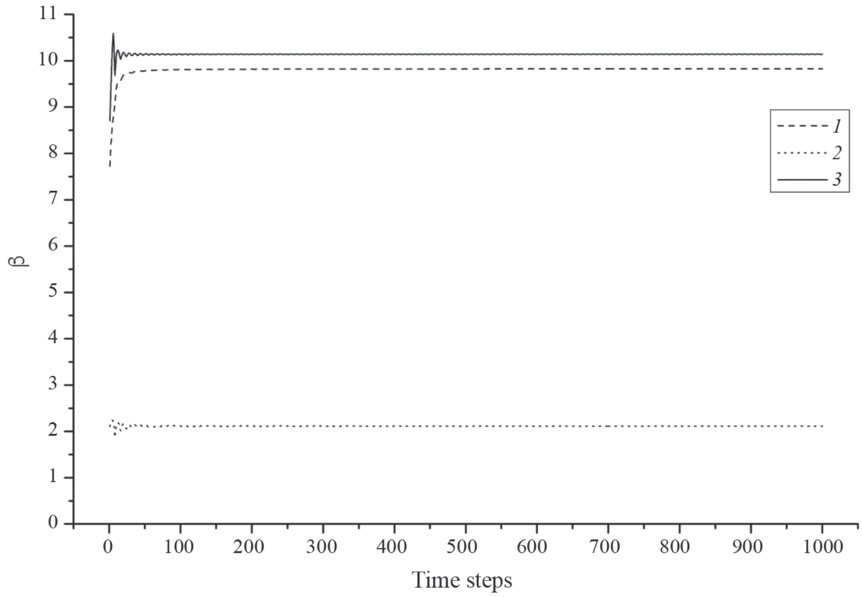

Фиг. 2. Поведение функции β(t) для схем расчета по формулам: *1* — (25), *2* — (35), *3* — (41) с течением времени.

На фиг. 1 приведены кривые $\log_{10}\mathrm{Er}(t, j)$, $j = 1, 2, 3$, для схем 10-го порядка точности (25), (35), (41). Как следует из графиков, среди трех представленных схем наилучший результат показывает схема (41). Несколько хуже результаты для схемы (25). Наихудший результат выдает формула (35).

На фиг. 2 представлены графики функции $\beta(t)$ для (25), (35), (41) в зависимости от времени. По результатам тестовых расчетов численная схема (35) не оправдала ожиданий. Среднее значение коэффициента Рунге равно 2. Наиболее вероятной причиной подобного поведения является относительно низкая точность численного решения нелинейной системы алгебраических уравнений (34). Несмотря на то что в формуле (41) содержится на 4 экспоненциальных множителя больше, чем в формуле (35), в силу ее точности и более простой практической реализации предпочтительнее использовать именно формулу (41) для численных расчетов.

5. ЗАКЛЮЧЕНИЕ

Представлена схема аппроксимации оператора эволюции (12) 10-го порядка точности с гораздо меньшим числом показателей, чем в формуле Ли–Троттера–Сузуки.

Рассмотрены две численные схемы с использованием операторных конструкций S_2 и S_4. Для обоих случаев определены системы нелинейных алгебраических уравнений для вычисления коэффициентов при операторных показателях. Решения систем для случая S_2 находились методом Монте-Карло с последующим варьированием коэффициентов для минимизации функции невязки, для случая S_4 система решалась в программном продукте Mathematica.

Были проведены тестовые расчеты для сравнения эффективности трех численных схем 10-го порядка: (25), (35) и (41). Результаты показывают хорошую относительную точность схемы (41). Крайне слабая эффективность схемы (35), по-видимому, объясняется недостаточной точностью найденных коэффициентов при экспонентах.

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ПРИЛОЖЕНИЕ А

Список необходимых слагаемых формулы Бэйкера–Кэмпбелла–Хаусдорфа

$$\exp\left(\frac{1}{2}X\right)\exp(Y)\exp\left(\frac{1}{2}X\right) = \exp\{Z(X,Y)\}:$$

Список необходимых слагаемых формулы Бэйкера-Кэмпбелла-Хаусдорфа $\exp\left(\frac{1}{2}X\right)\exp(Y)\exp\left(\frac{1}{2}X\right)= = \exp\left\{Z(X,Y)\right\}$:

$E_i = [E_{i'}, E_{i''}]$

$E_1 = N_X X$	$N_X = 2$	$E_{16} = [E_9, E_2] = N_{YXXXXY}\Xi_{YXXXXY}$	$N_{YXXXXY} = 0$
$E_2 = N_Y Y$	$N_Y = 1$	$E_{17} = [E_{10}, E_2] = N_{YXXXYY}\Xi_{YXXXYY}$	$N_{YXXXYY} = 0$
$E_3 = [E_2, E_1] = N_{YX}\Xi_{YX}$	$N_{YX} = 0$	$E_{18} = [E_{11}, E_2] = N_{YXXYYY}\Xi_{YXXYYY}$	$N_{YXXYYY} = 0$
$E_4 = [E_3, E_1] = N_{YXX}\Xi_{YXX}$	$N_{YXX} = -\frac{1}{6}$	$E_{19} = [E_{12}, E_2] = N_{YXYYYY}\Xi_{YXYYYY}$	$N_{YXYYYY} = 0$
$E_5 = [E_3, E_2] = N_{YXY}\Xi_{YXY}$	$N_{YXY} = -\frac{1}{6}$	$E_{20} = [E_6, E_3] = N_{YXXX,YX}\Xi_{YXXX,YX}$	$N_{YXXX,YX} = 0$
$E_6 = [E_4, E_1] = N_{YXXX}\Xi_{YXXX}$	$N_{YXXX} = 0$	$E_{21} = [E_7, E_3] = N_{YXXY,YX}\Xi_{YXXY,YX}$	$N_{YXXY,YX} = 0$
$E_7 = [E_4, E_2] = N_{YXXY}\Xi_{YXXY}$	$N_{YXXY} = 0$	$E_{22} = [E_8, E_3] = N_{YXYY,YX}\Xi_{YXYY,YX}$	$N_{YXYY,YX} = 0$
$E_8 = [E_5, E_2] = N_{YXYY}\Xi_{YXYY}$	$N_{YXYY} = 0$	$E_{23} = [E_5, E_4] = N_{YXY,YXX}\Xi_{YXY,YXX}$	$N_{YXY,YXX} = 0$
$E_9 = [E_6, E_1] = N_{YXXXX}\Xi_{YXXXX}$	$N_{YXXXX} = \frac{7}{360}$	$E_{24} = [E_{15}, E_1] = N_{YXXXXXX}\Xi_{YXXXXXX}$	$N_{YXXXXXX} = -\frac{31}{15120}$
$E_{10} = [E_6, E_2] = N_{YXXXY}\Xi_{YXXXY}$	$N_{YXXXY} = \frac{7}{180}$	$E_{25} = [E_{15}, E_2] = N_{YXXXXXY}\Xi_{YXXXXXY}$	$N_{YXXXXXY} = -\frac{31}{5040}$
$E_{11} = [E_7, E_2] = N_{YXXYY}\Xi_{YXXYY}$	$N_{YXXYY} = \frac{1}{45}$	$E_{26} = [E_{16}, E_2] = N_{YXXXXYY}\Xi_{YXXXXYY}$	$N_{YXXXXYY} = -\frac{13}{1890}$
$E_{12} = [E_8, E_2] = N_{YXYYY}\Xi_{YXYYY}$	$N_{YXYYY} = \frac{1}{360}$	$E_{27} = [E_{17}, E_2] = N_{YXXXYYY}\Xi_{YXXXYYY}$	$N_{YXXXYYY} = -\frac{53}{15120}$
$E_{13} = [E_4, E_3] = N_{YXX,YX}\Xi_{YXX,YX}$	$N_{YXX,YX} = \frac{1}{60}$	$E_{28} = [E_{18}, E_2] = N_{YXXYYYY}\Xi_{YXXYYYY}$	$N_{YXXYYYY} = -\frac{1}{1260}$
$E_{14} = [E_5, E_3] = N_{YXY,YX}\Xi_{YXY,YX}$	$N_{YXY,YX} = -\frac{1}{90}$	$E_{29} = [E_{19}, E_2] = N_{YXYYYYY}\Xi_{YXYYYYY}$	$N_{YXYYYYY} = -\frac{1}{15120}$
$E_{15} = [E_9, E_1] = N_{YXXXXX}\Xi_{YXXXXX}$	$N_{YXXXXX} = 0$		

Здесь

$\Xi_{ab} = [a,b]$

$\Xi_{abc} = [[a,b],c]$

$\Xi_{abcde} = [[[[a,b],c],d],e]$

$\Xi_{abc,de} = [[[a,b],c],[d,e]]$

$\Xi_{abs,de,fg} = [[[a,b],c],[d,e],[f,g]]$

$\Xi_{abcd,efg} = [[[[a,b],c],d],[[e,f],g]]$

ПРИЛОЖЕНИЕ В

Подставим $\Omega_0(\Theta_1,\Theta_3,\Theta_5,\Theta_7)$ и $\Omega_1(\Theta_1,\Theta_3,\Theta_5,\Theta_7)$ в формулу (31).

Введем следующие обозначения:

$$A_{1,0} = n_0;\ A_{3,0} = n_0^3;\ A_{5,0} = n_0^5;\ A_{7,0} = n_0^7;\ A_{9,0} = n_0^9. \tag{B.1}$$

Первые два слагаемых, $2\Xi_X + \Xi_Y$, можно записать как

$$\begin{gathered}\Omega_0 + 2\Omega_1 = \tau A_{1,0}\Theta_1 + \tau^3 A_{3,0}\Theta_3 + \\ + \tau^5 A_{5,0}\Theta_5 + \tau^7 A_{7,0}\Theta_7 + \tau^9 A_{9,0}\Theta_9 + \\ +2\tau n_1\Theta_1 + 2\tau^3 n_1^3\Theta_3 + 2\tau^5 n_1^5\Theta_5 + 2\tau^7 n_1^7\Theta_7 + 2\tau^9 n_1^9\Theta_9.\end{gathered} \tag{B.2.1}$$

Остальные слагаемые:

$$\begin{gathered}N_{YXX}\Xi_{\Omega_0\Omega_1\Omega_1} = \tau^5 N_{YXX}\left\{A_{1,0}n_1^4\Xi_{\Theta_1\Theta_3\Theta_1} + A_{3,0}n_1^2\Xi_{\Theta_3\Theta_1\Theta_1}\right\} + \\ +\tau^7 N_{YXX}\left\{A_{1,0}n_1^6\Xi_{\Theta_1\Theta_3\Theta_3} + A_{3,0}n_1^4\Xi_{\Theta_3\Theta_1\Theta_3} + A_{1,0}n_1^6\Xi_{\Theta_1\Theta_5\Theta_1} + A_{5,0}n_1^2\Xi_{\Theta_5\Theta_1\Theta_1}\right\} + \\ +\tau^9 N_{YXX}\left\{A_{1,0}n_1^8\Xi_{\Theta_1\Theta_3\Theta_5} + A_{3,0}n_1^6\Xi_{\Theta_3\Theta_1\Theta_5} + A_{1,0}n_1^8\Xi_{\Theta_1\Theta_5\Theta_3} + A_{5,0}n_1^4\Xi_{\Theta_5\Theta_1\Theta_3} + \right. \\ \left. + A_{3,0}n_1^6\Xi_{\Theta_3\Theta_5\Theta_1} + A_{5,0}n_1^4\Xi_{\Theta_5\Theta_3\Theta_1} + A_{1,0}n_1^8\Xi_{\Theta_1\Theta_7\Theta_1} + A_{7,0}n_1^2\Xi_{\Theta_7\Theta_1\Theta_1}\right\},\end{gathered} \tag{B.2.2}$$

$$\begin{gathered}N_{YXY}\Xi_{\Omega_0\Omega_1\Omega_0} = \tau^5 N_{YXY}\left\{\left(A_{1,0}\right)^2 n_1^3\Xi_{\Theta_1\Theta_3\Theta_1} + A_{1,0}A_{3,0}n_1\Xi_{\Theta_3\Theta_1\Theta_1}\right\} + \\ +\tau^7 N_{YXY}\left\{A_{1,0}A_{3,0}n_1^3\Xi_{\Theta_1\Theta_3\Theta_3} + \left(A_{3,0}\right)^2 n_1\Xi_{\Theta_3\Theta_1\Theta_3} + \left(A_{1,0}\right)^2 n_1^5\Xi_{\Theta_1\Theta_5\Theta_1} + A_{1,0}A_{5,0}n_1\Xi_{\Theta_5\Theta_1\Theta_1}\right\} + \\ +\tau^9 N_{YXY}\left\{A_{1,0}A_{5,0}n_1^3\Xi_{\Theta_1\Theta_3\Theta_5} + A_{3,0}A_{5,0}n_1\Xi_{\Theta_3\Theta_1\Theta_5} + A_{1,0}A_{3,0}n_1^5\Xi_{\Theta_1\Theta_5\Theta_3} + A_{3,0}A_{5,0}n_1\Xi_{\Theta_5\Theta_1\Theta_3} + \right. \\ \left. +A_{1,0}A_{3,0}n_1^5\Xi_{\Theta_3\Theta_5\Theta_1} + A_{1,0}A_{5,0}n_1^3\Xi_{\Theta_5\Theta_3\Theta_1} + \left(A_{1,0}\right)^2 n_1^7\Xi_{\Theta_1\Theta_7\Theta_1} + A_{1,0}A_{7,0}n_1\Xi_{\Theta_7\Theta_1\Theta_1}\right\},\end{gathered} \tag{B.2.3}$$

$$\begin{gathered}N_{YXXXX}\Xi_{\Omega_0\Omega_1\Omega_1\Omega_1\Omega_1} = \tau^7 N_{YXXXX}\left\{A_{1,0}n_1^6\Xi_{\Theta_1\Theta_3\Theta_1\Theta_1\Theta_1} + A_{3,0}n_1^4\Xi_{\Theta_3\Theta_1\Theta_1\Theta_1\Theta_1}\right\} + \\ +\tau^9 N_{YXXXX}\left\{A_{1,0}n_1^8\Xi_{\Theta_1\Theta_3\Theta_3\Theta_1\Theta_1} + A_{3,0}n_1^6\Xi_{\Theta_3\Theta_1\Theta_3\Theta_1\Theta_1} + A_{1,0}n_1^8\Xi_{\Theta_1\Theta_3\Theta_1\Theta_3\Theta_1} + \right. \\ +A_{3,0}n_1^6\Xi_{\Theta_3\Theta_1\Theta_1\Theta_3\Theta_1} + A_{1,0}n_1^8\Xi_{\Theta_1\Theta_3\Theta_1\Theta_1\Theta_3} + A_{3,0}n_1^6\Xi_{\Theta_3\Theta_1\Theta_1\Theta_1\Theta_3} + \\ \left. +A_{1,0}n_1^8\Xi_{\Theta_1\Theta_5\Theta_1\Theta_1\Theta_1} + A_{5,0}n_1^4\Xi_{\Theta_5\Theta_1\Theta_1\Theta_1\Theta_1}\right\},\end{gathered} \tag{B.2.4}$$

$$\begin{gathered}N_{YXXXY}\Xi_{\Omega_0\Omega_1\Omega_1\Omega_1\Omega_0} = \tau^7 N_{YXXXY}\left\{\left(A_{1,0}\right)^2 n_1^5\Xi_{\Theta_1\Theta_3\Theta_1\Theta_1\Theta_1} + A_{1,0}A_{3,0}n_1^3\Xi_{\Theta_3\Theta_1\Theta_1\Theta_1\Theta_1}\right\} + \\ +\tau^9 N_{YXXXY}\left\{\left(A_{1,0}\right)^2 n_1^7\Xi_{\Theta_1\Theta_3\Theta_3\Theta_1\Theta_1} + A_{1,0}A_{3,0}n_1^5\Xi_{\Theta_3\Theta_1\Theta_3\Theta_1\Theta_1} + \left(A_{1,0}\right)^2 n_1^7\Xi_{\Theta_1\Theta_3\Theta_1\Theta_3\Theta_1} + \right. \\ +A_{1,0}A_{3,0}n_1^5\Xi_{\Theta_3\Theta_1\Theta_1\Theta_3\Theta_1} + A_{1,0}A_{3,0}n_1^5\Xi_{\Theta_1\Theta_3\Theta_1\Theta_1\Theta_3} + \left(A_{3,0}\right)^2 n_1^3\Xi_{\Theta_3\Theta_1\Theta_1\Theta_1\Theta_3} + \\ \left. +\left(A_{1,0}\right)^2 n_1^7\Xi_{\Theta_1\Theta_5\Theta_1\Theta_1\Theta_1} + A_{1,0}A_{5,0}n_1^3\Xi_{\Theta_5\Theta_1\Theta_1\Theta_1\Theta_1}\right\},\end{gathered} \tag{B.2.5}$$

$$\begin{gathered}N_{YXXYY}\Xi_{\Omega_0\Omega_1\Omega_1\Omega_0\Omega_0} = \tau^7 N_{YXXYY}\left\{\left(A_{1,0}\right)^3 n_1^4\Xi_{\Theta_1\Theta_3\Theta_1\Theta_1\Theta_1} + A_{3,0}\left(A_{1,0}\right)^2 n_1^2\Xi_{\Theta_3\Theta_1\Theta_1\Theta_1\Theta_1}\right\} + \\ +\tau^9 N_{YXXYY}\left\{\left(A_{1,0}\right)^3 n_1^6\Xi_{\Theta_1\Theta_3\Theta_3\Theta_1\Theta_1} + A_{3,0}\left(A_{1,0}\right)^2 n_1^4\Xi_{\Theta_3\Theta_1\Theta_3\Theta_1\Theta_1} + A_{3,0}\left(A_{1,0}\right)^2 n_1^4\Xi_{\Theta_1\Theta_3\Theta_1\Theta_3\Theta_1} + \right.\end{gathered} \tag{B.2.6}$$

$$+A_{1,0}\left(A_{3,0}\right)^2 n_1^2\Xi_{\Theta_3\Theta_1\Theta_1\Theta_3\Theta_1}+A_{3,0}\left(A_{1,0}\right)^2 n_1^4\Xi_{\Theta_1\Theta_3\Theta_1\Theta_1\Theta_3}+A_{1,0}\left(A_{3,0}\right)^2 n_1^2\Xi_{\Theta_3\Theta_1\Theta_1\Theta_1\Theta_3}+$$
$$+\left(A_{1,0}\right)^3 n_1^6\,\Xi_{\Theta_1\Theta_5\Theta_1\Theta_1\Theta_1}+\left(A_{1,0}\right)^2 A_{5,0}n_1^2\Xi_{\Theta_5\Theta_1\Theta_1\Theta_1\Theta_1}\Big\},$$

$$\begin{gathered}N_{YXYYY}\Xi_{\Omega_0\Omega_1\Omega_0\Omega_0\Omega_0}=\tau^7 N_{YXYYY}\left\{\left(A_{1,0}\right)^4 n_1^3\Xi_{\Theta_1\Theta_3\Theta_1\Theta_1\Theta_1}+\left(A_{1,0}\right)^3 A_{3,0}n_1\Xi_{\Theta_3\Theta_1\Theta_1\Theta_1\Theta_1}\right\}+\\+\tau^9 N_{YXYYY}\Big\{\left(A_{1,0}\right)^3 A_{3,0}n_1^3\Xi_{\Theta_1\Theta_3\Theta_3\Theta_1\Theta_1}+\left(A_{1,0}\right)^2\left(A_{3,0}\right)^2 n_1\Xi_{\Theta_3\Theta_1\Theta_3\Theta_1\Theta_1}+\left(A_{1,0}\right)^3 A_{3,0}n_1^3\Xi_{\Theta_1\Theta_3\Theta_1\Theta_3\Theta_1}+\\+\left(A_{1,0}\right)^2\left(A_{3,0}\right)^2 n_1\Xi_{\Theta_3\Theta_1\Theta_1\Theta_3\Theta_1}+\left(A_{1,0}\right)^3 A_{3,0}n_1^3\Xi_{\Theta_1\Theta_3\Theta_1\Theta_1\Theta_3}+\left(A_{1,0}\right)^2\left(A_{3,0}\right)^2 n_1\Xi_{\Theta_3\Theta_1\Theta_1\Theta_1\Theta_3}+\\+\left(A_{1,0}\right)^4 n_1^5\,\Xi_{\Theta_1\Theta_5\Theta_1\Theta_1\Theta_1}+\left(A_{1,0}\right)^3 A_{5,0}n_1\Xi_{\Theta_5\Theta_1\Theta_1\Theta_1\Theta_1}\Big\},\end{gathered}\tag{B.2.7}$$

$$\begin{gathered}N_{YXX,YX}\Xi_{\Omega_0\Omega_1\Omega_1,\Omega_0\Omega_1}=N_{YXX,YX}\tau^9\Big\{\left(A_{1,0}\right)^2 n_1^7\Xi_{\Theta_1\Theta_3\Theta_1,\Theta_1\Theta_3}+A_{1,0}A_{3,0}n_1^5\Xi_{\Theta_3\Theta_1\Theta_1,\Theta_1\Theta_3}\\+A_{1,0}A_{3,0}n_1^5\Xi_{\Theta_1\Theta_3\Theta_1,\Theta_3\Theta_1}+\left(A_{3,0}\right)^2 n_1^3\Xi_{\Theta_3\Theta_1\Theta_1,\Theta_3\Theta_1}\Big\},\end{gathered}\tag{B.2.8}$$

$$\begin{gathered}N_{YXY,YX}\Xi_{\Omega_0\Omega_1\Omega_0,\Omega_0\Omega_1}=N_{YXY,YX}\tau^9\Big\{\left(A_{1,0}\right)^3 n_1^6\Xi_{\Theta_1\Theta_3\Theta_1,\Theta_1\Theta_3}+\left(A_{1,0}\right)^2 A_{3,0}n_1^4\Xi_{\Theta_3\Theta_1\Theta_1,\Theta_1\Theta_3}\\+\left(A_{1,0}\right)^2 A_{3,0}n_1^4\Xi_{\Theta_1\Theta_3\Theta_1,\Theta_3\Theta_1}+A_{1,0}\left(A_{3,0}\right)^2 n_1^2\Xi_{\Theta_3\Theta_1\Theta_1,\Theta_3\Theta_1}\Big\},\end{gathered}\tag{B.2.9}$$

$$N_{YXXXXXX}\Xi_{\Omega_0\Omega_1\Omega_1\Omega_1\Omega_1\Omega_1\Omega_1}=\tau^9 N_{YXXXXXX}\left\{A_{1,0}n_1^8\Xi_{\Theta_1\Theta_3\Theta_1\Theta_1\Theta_1\Theta_1\Theta_1}+A_{3,0}n_1^6\Xi_{\Theta_3\Theta_1\Theta_1\Theta_1\Theta_1\Theta_1\Theta_1}\right\},\tag{B.2.10}$$

$$N_{YXXXXXY}\Xi_{\Omega_0\Omega_1\Omega_1\Omega_1\Omega_1\Omega_1\Omega_0}=\tau^9 N_{YXXXXXY}\left\{\left(A_{1,0}\right)^2 n_1^7\Xi_{\Theta_1\Theta_3\Theta_1\Theta_1\Theta_1\Theta_1\Theta_1}+A_{3,0}A_{1,0}n_1^5\Xi_{\Theta_3\Theta_1\Theta_1\Theta_1\Theta_1\Theta_1\Theta_1}\right\},\tag{B.2.11}$$

$$N_{YXXXXYY}\Xi_{\Omega_0\Omega_1\Omega_1\Omega_1\Omega_1\Omega_0\Omega_0}=\tau^9 N_{YXXXXYY}\left\{\left(A_{1,0}\right)^3 n_1^6\Xi_{\Theta_1\Theta_3\Theta_1\Theta_1\Theta_1\Theta_1\Theta_1}+A_{3,0}\left(A_{1,0}\right)^2 n_1^4\Xi_{\Theta_3\Theta_1\Theta_1\Theta_1\Theta_1\Theta_1\Theta_1}\right\},\tag{B.2.12}$$

$$N_{YXXXYYY}\Xi_{\Omega_0\Omega_1\Omega_1\Omega_1\Omega_0\Omega_0\Omega_0}=\tau^9 N_{YXXXYYY}\left\{\left(A_{1,0}\right)^4 n_1^5\Xi_{\Theta_1\Theta_3\Theta_1\Theta_1\Theta_1\Theta_1\Theta_1}+A_{3,0}\left(A_{1,0}\right)^3 n_1^3\Xi_{\Theta_3\Theta_1\Theta_1\Theta_1\Theta_1\Theta_1\Theta_1}\right\},\tag{B.2.13}$$

$$N_{YXXYYYY}\Xi_{\Omega_0\Omega_1\Omega_1\Omega_0\Omega_0\Omega_0\Omega_0}=\tau^9 N_{YXXYYYY}\left\{\left(A_{1,0}\right)^5 n_1^4\Xi_{\Theta_1\Theta_3\Theta_1\Theta_1\Theta_1\Theta_1\Theta_1}+A_{3,0}\left(A_{1,0}\right)^4 n_1^2\Xi_{\Theta_3\Theta_1\Theta_1\Theta_1\Theta_1\Theta_1\Theta_1}\right\},\tag{B.2.14}$$

$$N_{YXYYYYY}\Xi_{\Omega_0\Omega_1\Omega_0\Omega_0\Omega_0\Omega_0\Omega_0}=\tau^9 N_{YXYYYYY}\left\{\left(A_{1,0}\right)^6 n_1^3\Xi_{\Theta_1\Theta_3\Theta_1\Theta_1\Theta_1\Theta_1\Theta_1}+A_{3,0}\left(A_{1,0}\right)^5 n_1\Xi_{\Theta_3\Theta_1\Theta_1\Theta_1\Theta_1\Theta_1\Theta_1}\right\}.\tag{B.2.15}$$

В итоге выражение $\Omega^{(1)}$ можно представить в виде

$$\begin{gathered}\Omega^{(1)}=\tau A_{1,1}\Xi_{\Theta_1}+\tau^3 A_{3,1}\Xi_{\Theta_3}+\tau^5\left(A_{5,1}\Xi_{\Theta_5}+B_{5,1}\Xi_{\Theta_3\Theta_1\Theta_1}\right)\\+\tau^7\left(A_{7,1}\Xi_{\Theta_7}+B_{7,1}\Xi_{\Theta_5\Theta_1\Theta_1}+B_{7,1}^3\Xi_{\Theta_3\Theta_1\Theta_3}+C_{7,1}\Xi_{\Theta_3\Theta_1\Theta_1\Theta_1\Theta_1}\right)\\+\tau^9\Big(A_{9,1}\Xi_{\Theta_9}+B_{9,1}\Xi_{\Theta_7\Theta_1\Theta_1}+B_{9,1}^1\Xi_{\Theta_5\Theta_3\Theta_1}+B_{9,1}^3\Xi_{\Theta_5\Theta_1\Theta_3}+B_{9,1}^5\Xi_{\Theta_3\Theta_1\Theta_5}+C_{9,1}\Xi_{\Theta_5\Theta_1\Theta_1\Theta_1\Theta_1}\\+C_{9,1}^{311}\Xi_{\Theta_3\Theta_1\Theta_3\Theta_1\Theta_1}+C_{9,1}^{131}\Xi_{\Theta_3\Theta_1\Theta_1\Theta_3\Theta_1}+C_{9,1}^{113}\Xi_{\Theta_3\Theta_1\Theta_1\Theta_1\Theta_3}+E_{9,1}\Xi_{\Theta_3\Theta_1\Theta_1,\Theta_3\Theta_1}+D_{9,1}\Xi_{\Theta_3\Theta_1\Theta_1\Theta_1\Theta_1\Theta_1\Theta_1}\Big).\end{gathered}\tag{B.3}$$

Здесь

$$A_{1,1}\equiv A_{1,0}+2n_1,\tag{B.4.1}$$

$$A_{3,1}\equiv A_{3,0}+2n_1^3,\tag{B.4.2}$$

$$A_{5,1} \equiv A_{5,0} + 2n_1^5, \tag{B.4.3}$$

$$A_{7,1} \equiv A_{7,0} + 2n_1^7, \tag{B.4.4}$$

$$A_{9,1} \equiv A_{9,0} + 2n_1^9, \tag{B.4.5}$$

$$B_{5,1} = N_{YXX}\left\{A_{3,0}n_1^2 - A_{1,0}n_1^4\right\} + N_{YXY}\left\{A_{1,0}A_{3,0}n_1 - \left(A_{1,0}\right)^2 n_1^3\right\}, \tag{B.4.6}$$

$$B_{7,1} = N_{YXX}\left\{A_{5,0}n_1^2 - A_{1,0}n_1^6\right\} + N_{YXY}\left\{A_{1,0}A_{5,0}n_1 - \left(A_{1,0}\right)^2 n_1^5\right\}, \tag{B.4.7}$$

$$B_{7,1}^3 = N_{YXX}\left\{A_{3,0}n_1^4 - A_{1,0}n_1^6\right\} + N_{YXY}\left\{\left(A_{3,0}\right)^2 n_1 - A_{1,0}A_{3,0}n_1^3\right\}, \tag{B.4.8}$$

$$B_{9,1} = N_{YXX}\left\{A_{7,0}n_1^2 - A_{1,0}n_1^8\right\} + N_{YXY}\left\{A_{1,0}A_{7,0}n_1 - \left(A_{1,0}\right)^2 n_1^7\right\}, \tag{B.4.9}$$

$$B_{9,1}^1 = N_{YXX}\left\{A_{5,0}n_1^4 - A_{3,0}n_1^6\right\} + N_{YXY}\left\{A_{1,0}A_{5,0}n_1^3 - A_{1,0}A_{3,0}n_1^5\right\}, \tag{B.4.10}$$

$$B_{9,1}^3 = N_{YXX}\left\{A_{5,0}n_1^4 - A_{1,0}n_1^8\right\} + N_{YXY}\left\{A_{3,0}A_{5,0}n_1 - A_{1,0}A_{3,0}n_1^5\right\}, \tag{B.4.11}$$

$$B_{9,1}^5 = N_{YXX}\left\{A_{3,0}n_1^6 - A_{1,0}n_1^8\right\} + N_{YXY}\left\{A_{3,0}A_{5,0}n_1 - A_{1,0}A_{5,0}n_1^3\right\}, \tag{B.4.12}$$

$$\begin{aligned} C_{7,1} &= N_{YXXXX}\left\{A_{3,0}n_1^4 - A_{1,0}n_1^6\right\} + N_{YXXXY}\left\{A_{1,0}A_{3,0}n_1^3 - \left(A_{1,0}\right)^2 n_1^5\right\} \\ &+N_{YXXYY}\left\{A_{3,0}\left(A_{1,0}\right)^2 n_1^2 - \left(A_{1,0}\right)^3 n_1^4\right\} + N_{YXYYY}\left\{\left(A_{1,0}\right)^3 A_{3,0}n_1 - \left(A_{1,0}\right)^4 n_1^3\right\}, \end{aligned} \tag{B.4.13}$$

$$\begin{aligned} C_{9,1} &= N_{YXXXX}\left\{A_{5,0}n_1^4 - A_{1,0}n_1^8\right\} + N_{YXXXY}\left\{A_{1,0}A_{5,0}n_1^3 - \left(A_{1,0}\right)^2 n_1^7\right\} \\ &+N_{YXXYY}\left\{\left(A_{1,0}\right)^2 A_{5,0}n_1^2 - \left(A_{1,0}\right)^3 n_1^6\right\} + N_{YXYYY}\left\{\left(A_{1,0}\right)^3 A_{5,0}n_1 - \left(A_{1,0}\right)^4 n_1^5\right\}, \end{aligned} \tag{B.4.14}$$

$$\begin{aligned} C_{9,1}^{311} &= N_{YXXXX}\left\{A_{3,0}n_1^6 - A_{1,0}n_1^8\right\} + N_{YXXXY}\left\{A_{1,0}A_{3,0}n_1^5 - \left(A_{1,0}\right)^2 n_1^7\right\} \\ &+N_{YXXYY}\left\{A_{3,0}\left(A_{1,0}\right)^2 n_1^4 - \left(A_{1,0}\right)^3 n_1^6\right\} + N_{YXYYY}\left\{\left(A_{1,0}\right)^2\left(A_{3,0}\right)^2 n_1 - \left(A_{1,0}\right)^3 A_{3,0}n_1^3\right\}, \end{aligned} \tag{B.4.15}$$

$$\begin{aligned} C_{9,1}^{131} &= N_{YXXXX}\left\{A_{3,0}n_1^6 - A_{1,0}n_1^8\right\} + N_{YXXXY}\left\{A_{1,0}A_{3,0}n_1^5 - \left(A_{1,0}\right)^2 n_1^7\right\} \\ &+N_{YXXYY}\left\{A_{1,0}\left(A_{3,0}\right)^2 n_1^2 - A_{3,0}\left(A_{1,0}\right)^2 n_1^4\right\} + N_{YXYYY}\left\{\left(A_{1,0}\right)^2\left(A_{3,0}\right)^2 n_1 - \left(A_{1,0}\right)^3 A_{3,0}n_1^3\right\}, \end{aligned} \tag{B.4.16}$$

$$\begin{aligned} C_{9,1}^{113} &= N_{YXXXX}\left\{A_{3,0}n_1^6 - A_{1,0}n_1^8\right\} + N_{YXXXY}\left\{\left(A_{3,0}\right)^2 n_1^3 - A_{1,0}A_{3,0}n_1^5\right\} \\ &+N_{YXXYY}\left\{A_{1,0}\left(A_{3,0}\right)^2 n_1^2 - A_{3,0}\left(A_{1,0}\right)^2 n_1^4\right\} + N_{YXYYY}\left\{\left(A_{1,0}\right)^2\left(A_{3,0}\right)^2 n_1 - \left(A_{1,0}\right)^3 A_{3,0}n_1^3\right\}, \end{aligned} \tag{B.4.17}$$

$$E_{9,1} = N_{YXX,YX}\left\{\left(A_{3,0}\right)^2 n_1^3 + \left(A_{1,0}\right)^2 n_1^7 - 2A_{1,0}A_{3,0}n_1^5\right\}$$
$$+N_{YXY,YX}\left\{A_{1,0}\left(A_{3,0}\right)^2 n_1^2 + \left(A_{1,0}\right)^3 n_1^6 - 2\left(A_{1,0}\right)^2 A_{3,0}n_1^4\right\}, \quad (B.4.18)$$

$$D_{9,1} = N_{YXXXXXX}\left\{A_{3,0}n_1^6 - A_{1,0}n_1^8\right\} + N_{YXXXXXY}\left\{A_{3,0}A_{1,0}n_1^5 - \left(A_{1,0}\right)^2 n_1^7\right\}$$
$$+N_{YXXXXYY}\left\{A_{3,0}\left(A_{1,0}\right)^2 n_1^4 - \left(A_{1,0}\right)^3 n_1^6\right\} + N_{YXXXYYY}\left\{A_{3,0}\left(A_{1,0}\right)^3 n_1^3 - \left(A_{1,0}\right)^4 n_1^5\right\} \quad (B.4.19)$$
$$+N_{YXXYYYY}\left\{A_{3,0}\left(A_{1,0}\right)^4 n_1^2 - \left(A_{1,0}\right)^5 n_1^4\right\} + N_{YXYYYYY}\left\{A_{3,0}\left(A_{1,0}\right)^5 n_1 - \left(A_{1,0}\right)^6 n_1^3\right\}.$$

Подставим теперь $\Omega^{(1)}\left(\Theta_1,\Theta_3,\Theta_5,\Theta_7\right)$ и $\Omega_2\left(\Theta_1,\Theta_3,\Theta_5,\Theta_7\right)$ в формулу (31).

Первые два слагаемых, $2\Xi_X + \Xi_Y$, можно записать в виде

$$\Omega^{(1)} + 2\Omega_2 = \tau A_{1,1}\Theta_1 + \tau^3 A_{3,1}\Theta_3 + \tau^5 A_{5,1}\Theta_5 + \tau^7 A_{7,1}\Theta_7 + \tau^9 A_{9,1}\Theta_9 +$$
$$+2\tau n_2\Theta_1 + 2\tau^3 n_2^3\Theta_3 + 2\tau^5 n_2^5\Theta_5 + 2\tau^7 n_2^7\Theta_7 + 2\tau^9 n_2^9\Theta_9. \quad (B.5.1)$$

Обозначим $G\left(\Theta_1,\Theta_3,\Theta_5,\Theta_7\right)$ как вклад, схожий по форме для предыдущего шага. Тогда выражения с новыми слагаемыми запишутся как

$$N_{YXX}\Xi_{\Omega^{(1)}\Omega_2\Omega_2} = G\left(\Theta_1,\Theta_3,\Theta_5,\Theta_7\right) + \tau^7 N_{YXX}B_{5,1}n_2^2\Xi_{\Theta_3\Theta_1\Theta_1\Theta_1\Theta_1} + \tau^9 N_{YXX}B_{5,1}n_2^4\Xi_{\Theta_3\Theta_1\Theta_1\Theta_3\Theta_1}$$
$$+\tau^9 N_{YXX}B_{5,1}n_2^4\Xi_{\Theta_3\Theta_1\Theta_1\Theta_1\Theta_3} + \tau^9 N_{YXX}B_{7,1}n_2^2\Xi_{\Theta_5\Theta_1\Theta_1\Theta_1\Theta_1} + \tau^9 N_{YXX}B_{7,1}^3 n_2^2\Xi_{\Theta_3\Theta_1\Theta_3\Theta_1\Theta_1} \quad (B.5.2)$$
$$+\tau^9 N_{YXX}C_{7,1}n_2^2\Xi_{\Theta_3\Theta_1\Theta_1\Theta_1\Theta_1\Theta_1\Theta_1},$$

$$N_{YXY}\Xi_{\Omega^{(1)}\Omega_2\Omega^{(1)}} = G\left(\Theta_1,\Theta_3,\Theta_5,\Theta_7\right) + \tau^7 N_{YXY}B_{5,1}A_{1,1}n_2\Xi_{\Theta_3\Theta_1\Theta_1\Theta_1\Theta_1} + \tau^9 N_{YXY}B_{5,1}A_{1,1}n_2^3\Xi_{\Theta_3\Theta_1\Theta_1\Theta_3\Theta_1}$$
$$+\tau^9 N_{YXY}B_{5,1}A_{3,1}n_2\Xi_{\Theta_3\Theta_1\Theta_1\Theta_1\Theta_3} + \tau^9 N_{YXY}B_{7,1}A_{1,1}n_2\Xi_{\Theta_5\Theta_1\Theta_1\Theta_1\Theta_1} + \tau^9 N_{YXY}B_{7,1}^3 A_{1,1}n_2\Xi_{\Theta_3\Theta_1\Theta_3\Theta_1\Theta_1} \quad (B.5.3)$$
$$+\tau^9 N_{YXY}C_{7,1}A_{1,1}n_2\Xi_{\Theta_3\Theta_1\Theta_1\Theta_1\Theta_1\Theta_1\Theta_1} + N_{YXY}\tau^9 A_{1,1}n_2^3 B_{5,1}\Xi_{\Theta_1\Theta_3,\Theta_3\Theta_1\Theta_1} + N_{YXY}\tau^9 A_{3,1}n_2 B_{5,1}\Xi_{\Theta_3\Theta_1,\Theta_3\Theta_1\Theta_1},$$

$$N_{YXXXX}\Xi_{\Omega^{(1)}\Omega_1\Omega_1\Omega_1\Omega_1} = G\left(\Theta_1,\Theta_3,\Theta_5\right) + \tau^9 N_{YXXXX}B_{5,1}n_2^4\Xi_{\Theta_3\Theta_1\Theta_1\Theta_1\Theta_1\Theta_1\Theta_1}, \quad (B.5.4)$$

$$N_{YXXXY}\Xi_{\Omega^{(1)}\Omega_1\Omega_1\Omega_1\Omega^{(1)}} = G\left(\Theta_1,\Theta_3,\Theta_5\right) + \tau^9 N_{YXXXY}B_{5,1}A_{1,1}n_2^3\Xi_{\Theta_3\Theta_1\Theta_1\Theta_1\Theta_1\Theta_1\Theta_1}, \quad (B.5.5)$$

$$N_{YXXYY}\Xi_{\Omega^{(1)}\Omega_1\Omega_1\Omega^{(1)}\Omega^{(1)}} = G\left(\Theta_1,\Theta_3,\Theta_5\right) + \tau^9 N_{YXXYY}B_{5,1}\left(A_{1,1}\right)^2 n_2^2\Xi_{\Theta_3\Theta_1\Theta_1\Theta_1\Theta_1\Theta_1\Theta_1}, \quad (B.5.6)$$

$$N_{YXYYY}\Xi_{\Omega^{(1)}\Omega_1\Omega^{(1)}\Omega^{(1)}\Omega^{(1)}} = G\left(\Theta_1,\Theta_3,\Theta_5\right) + \tau^9 N_{YXYYY}B_{5,1}\left(A_{1,1}\right)^3 n_2\Xi_{\Theta_3\Theta_1\Theta_1\Theta_1\Theta_1\Theta_1\Theta_1}. \quad (B.5.7)$$

Подстановка функций более высокого порядка $\Omega^{(m)}$ не обнаружит новых слагаемых. Поэтому окончательный результат может быть записан следующей рекуррентной формулой:

$$\Omega^{(m)} = \tau A_{1,m}\Xi_{\Theta_1} + \tau^3 A_{3,m}\Xi_{\Theta_3} + \tau^5\left(A_{5,m}\Xi_{\Theta_5} + B_{5,m}\Xi_{\Theta_3\Theta_1\Theta_1}\right)$$
$$+\tau^7\left(A_{7,m}\Xi_{\Theta_7} + B_{7,m}\Xi_{\Theta_5\Theta_1\Theta_1} + B_{7,m}^3\Xi_{\Theta_3\Theta_1\Theta_3} + C_{7,m}\Xi_{\Theta_3\Theta_1\Theta_1\Theta_1\Theta_1}\right)$$
$$+\tau^9\left(A_{9,m}\Xi_{\Theta_9} + B_{9,m}\Xi_{\Theta_7\Theta_1\Theta_1} + B_{9,m}^1\Xi_{\Theta_5\Theta_3\Theta_1} + B_{9,m}^3\Xi_{\Theta_5\Theta_1\Theta_3} + B_{9,m}^5\Xi_{\Theta_3\Theta_1\Theta_5} + C_{9,m}\Xi_{\Theta_5\Theta_1\Theta_1\Theta_1\Theta_1}\right. \quad (B.6)$$
$$\left.+C_{9,m}^{311}\Xi_{\Theta_3\Theta_1\Theta_3\Theta_1\Theta_1} + C_{9,m}^{131}\Xi_{\Theta_3\Theta_1\Theta_1\Theta_3\Theta_1} + C_{9,m}^{113}\Xi_{\Theta_3\Theta_1\Theta_1\Theta_1\Theta_3} + E_{9,m}\Xi_{\Theta_3\Theta_1\Theta_1,\Theta_3\Theta_1} + D_{9,m}\Xi_{\Theta_3\Theta_1\Theta_1\Theta_1\Theta_1\Theta_1\Theta_1}\right).$$

Здесь

$$A_{1,m}=A_{1,m-1}+2n_m, \tag{B.7.1}$$

$$A_{3,m}=A_{3,m-1}+2n_m^3, \tag{B.7.2}$$

$$A_{5,m}=A_{5,m-1}+2n_m^5, \tag{B.7.3}$$

$$A_{7,m}=A_{7,m-1}+2n_m^7, \tag{B.7.4}$$

$$A_{9,m}=A_{9,m-1}+2n_m^9, \tag{B.7.5}$$

$$B_{5,m}=B_{5,m-1}+N_{YXX}\left\{A_{3,m-1}n_m^2-A_{1,m-1}n_m^4\right\}+N_{YXY}\left\{A_{1,m-1}A_{3,m-1}n_m-\left(A_{1,m-1}\right)^2n_m^3\right\}, \tag{B.7.6}$$

$$B_{7,m}=B_{7,m-1}+N_{YXX}\left\{A_{5,m-1}n_m^2-A_{1,m-1}n_m^6\right\}+N_{YXY}\left\{A_{1,m-1}A_{5,m-1}n_m-\left(A_{1,m-1}\right)^2n_m^5\right\}, \tag{B.7.7}$$

$$B_{7,m}^3=B_{7,m-1}^3+N_{YXX}\left\{A_{3,m-1}n_m^4-A_{1,m-1}n_m^6\right\}+N_{YXY}\left\{\left(A_{3,m-1}\right)^2n_m-A_{1,m-1}A_{3,m-1}n_m^3\right\}, \tag{B.7.8}$$

$$B_{9,m}=B_{9,m-1}+N_{YXX}\left\{A_{7,m-1}n_m^2-A_{1,m-1}n_m^8\right\}+N_{YXY}\left\{A_{1,m-1}A_{7,m-1}n_m-\left(A_{1,m-1}\right)^2n_m^7\right\}, \tag{B.7.9}$$

$$B_{9,m}^1=B_{9,m-1}^1+N_{YXX}\left\{A_{5,m-1}n_m^4-A_{3,m-1}n_m^6\right\}+N_{YXY}\left\{A_{1,m-1}A_{5,m-1}n_m^3-A_{1,m-1}A_{3,m-1}n_m^5\right\}, \tag{B.7.10}$$

$$B_{9,m}^3=B_{9,m-1}^3+N_{YXX}\left\{A_{5,m-1}n_m^4-A_{1,m-1}n_m^8\right\}+N_{YXY}\left\{A_{3,m-1}A_{5,m-1}n_m-A_{1,m-1}A_{3,m-1}n_m^5\right\}, \tag{B.7.11}$$

$$B_{9,m}^5=B_{9,m-1}^5+N_{YXX}\left\{A_{3,m-1}n_m^6-A_{1,m-1}n_m^8\right\}+N_{YXY}\left\{A_{3,m-1}A_{5,m-1}n_m-A_{1,m-1}A_{5,m-1}n_m^3\right\}, \tag{B.7.12}$$

$$\begin{gathered}C_{7,m}=C_{7,m-1}+N_{YXXXX}\left\{A_{3,m-1}n_m^4-A_{1,m-1}n_m^6\right\}+N_{YXXXY}\left\{A_{1,m-1}A_{3,m-1}n_m^3-\left(A_{1,m-1}\right)^2n_m^5\right\}\\+N_{YXXYY}\left\{A_{3,m-1}\left(A_{1,m-1}\right)^2n_m^2-\left(A_{1,m-1}\right)^3n_m^4\right\}+N_{YXYYY}\left\{\left(A_{1,m-1}\right)^3A_{3,m-1}n_m-\left(A_{1,m-1}\right)^4n_m^3\right\}\\+N_{YXX}B_{5,m-1}n_m^2+N_{YXY}B_{5,m-1}A_{1,m-1}n_m,\end{gathered} \tag{B.7.13}$$

$$\begin{gathered}C_{9,m}=C_{9,m-1}+N_{YXXXX}\left\{A_{5,m-1}n_m^4-A_{1,m-1}n_m^8\right\}+N_{YXXXY}\left\{A_{1,m-1}A_{5,m-1}n_m^3-\left(A_{1,m-1}\right)^2n_m^7\right\}\\+N_{YXXYY}\left\{\left(A_{1,m-1}\right)^2A_{5,m-1}n_m^2-\left(A_{1,m-1}\right)^3n_m^6\right\}+N_{YXYYY}\left\{\left(A_{1,m-1}\right)^3A_{5,m-1}n_m-\left(A_{1,m-1}\right)^4n_m^5\right\}\\+N_{YXX}B_{7,m-1}n_m^2+N_{YXY}B_{7,m-1}A_{1,m-1}n_m,\end{gathered} \tag{B.7.14}$$

$$\begin{gathered}C_{9,m}^{311}=C_{9,m-1}^{311}+N_{YXXXX}\left\{A_{3,m-1}n_m^6-A_{1,m-1}n_m^8\right\}+N_{YXXXY}\left\{A_{1,m-1}A_{3,m-1}n_m^5-\left(A_{1,m-1}\right)^2n_m^7\right\}\\+N_{YXXYY}\left\{A_{3,m-1}\left(A_{1,m-1}\right)^2n_m^4-\left(A_{1,m-1}\right)^3n_m^6\right\}+N_{YXYYY}\left\{\left(A_{1,m-1}\right)^2\left(A_{3,m-1}\right)^2n_m-\left(A_{1,m-1}\right)^3A_{3,m-1}n_m^3\right\}\\+N_{YXX}B_{7,m-1}^3n_m^2+N_{YXY}B_{7,m-1}^3A_{1,m-1}n_m,\end{gathered} \tag{B.7.15}$$

$$\begin{gathered}C_{9,m}^{131}=C_{9,m-1}^{131}+N_{YXXXX}\left\{A_{3,m-1}n_m^6-A_{1,m-1}n_m^8\right\}+N_{YXXXY}\left\{A_{1,m-1}A_{3,m-1}n_m^5-\left(A_{1,m-1}\right)^2n_m^7\right\}\\+N_{YXXYY}\left\{A_{1,m-1}\left(A_{3,m-1}\right)^2n_m^2-A_{3,m-1}\left(A_{1,m-1}\right)^2n_m^4\right\}\\+N_{YXYYY}\left\{\left(A_{1,m-1}\right)^2\left(A_{3,m-1}\right)^2n_m-\left(A_{1,m-1}\right)^3A_{3,m-1}n_m^3\right\}+N_{YXX}B_{5,m-1}n_m^4+N_{YXY}B_{5,m-1}A_{1,m-1}n_m^3,\end{gathered} \tag{B.7.16}$$

$$C_{9,m}^{113}=C_{9,m-1}^{113}+N_{YXXXX}\left\{A_{3,m-1}n_m^6-A_{1,m-1}n_m^8\right\}+N_{YXXXY}\left\{\left(A_{3,m-1}\right)^2n_m^3-A_{1,m-1}A_{3,m-1}n_m^5\right\}$$
$$+N_{YXXYY}\left\{A_{1,m-1}\left(A_{3,m-1}\right)^2n_m^2-A_{3,m-1}\left(A_{1,m-1}\right)^2n_m^4\right\} \qquad \text{(B.7.17)}$$
$$+N_{YXYYY}\left\{\left(A_{1,m-1}\right)^2\left(A_{3,m-1}\right)^2n_m-\left(A_{1,m-1}\right)^3A_{3,m-1}n_m^3\right\}+N_{YXX}B_{5,m-1}n_m^4+N_{YXY}B_{5,m-1}A_{3,m-1}n_m,$$

$$E_{9,m}=E_{9,m-1}+N_{YXX,YX}\left\{\left(A_{3,m-1}\right)^2n_m^3+\left(A_{1,m-1}\right)^2n_m^7-2A_{1,m-1}A_{3,m-1}n_m^5\right\}$$
$$+N_{YXY,YX}\left\{A_{1,m-1}\left(A_{3,m-1}\right)^2n_m^2+\left(A_{1,m-1}\right)^3n_m^6-2\left(A_{1,m-1}\right)^2A_{3,m-1}n_m^4\right\} \qquad \text{(B.7.18)}$$
$$+N_{YXY}A_{1,m-1}B_{5,m-1}n_m^3-N_{YXY}A_{3,m-1}B_{5,m-1}n_m,$$

$$D_{9,m}=D_{9,m-1}+N_{YXXXXXX}\left\{A_{3,m-1}n_m^6-A_{1,m-1}n_m^8\right\}+N_{YXXXXXY}\left\{A_{3,m-1}A_{1,m-1}n_m^5-\left(A_{1,m-1}\right)^2n_m^7\right\}$$
$$+N_{YXXXXYY}\left\{A_{3,m-1}\left(A_{1,m-1}\right)^2n_m^4-\left(A_{1,m-1}\right)^3n_m^6\right\}+N_{YXXXYYY}\left\{A_{3,m-1}\left(A_{1,m-1}\right)^3n_m^3-\left(A_{1,m-1}\right)^4n_m^5\right\}$$
$$+N_{YXXYYYY}\left\{A_{3,m-1}\left(A_{1,m-1}\right)^4n_m^2-\left(A_{1,m-1}\right)^5n_m^4\right\}+N_{YXYYYYY}\left\{A_{3,m-1}\left(A_{1,m-1}\right)^5n_m-\left(A_{1,m-1}\right)^6n_m^3\right\} \qquad \text{(B.7.19)}$$
$$+N_{YXX}C_{7,m-1}n_m^2+N_{YXY}C_{7,m-1}A_{1,m-1}n_m+N_{YXXXX}B_{5,m-1}n_m^4+N_{YXXXY}B_{5,m-1}A_{1,m-1}n_m^3$$
$$+N_{YXXYY}B_{5,m-1}\left(A_{1,m-1}\right)^2n_m^2+N_{YXYYY}B_{5,m-1}\left(A_{1,m-1}\right)^3n_m,$$

ПРИЛОЖЕНИЕ C

Подставим $\Omega_0\left(\Theta_1,\tilde\Theta_5,\tilde\Theta_7,\tilde\Theta_9\right)$ и $\Omega_1\left(\Theta_1,\tilde\Theta_5,\tilde\Theta_7,\tilde\Theta_9\right)$ в формулу (31).

Введем следующие обозначения:

$$A_{1,0}=n_0;\ A_{3,0}=n_0^3;\ A_{5,0}=n_0^5;\ A_{7,0}=n_0^7;\ A_{9,0}=n_0^9. \qquad \text{(C.1)}$$

Первые два слагаемых, $2\Xi_X+\Xi_Y$, будут записаны как

$$\Omega_0+2\Omega_1=\tau A_{1,0}\Theta_1+\tau^3A_{3,0}\tilde\Theta_3+\tau^5A_{5,0}\tilde\Theta_5+\tau^7A_{7,0}\tilde\Theta_7+\tau^9A_{9,0}\tilde\Theta_9+$$
$$+2\tau n_1\Theta_1+2\tau^3n_1^3\tilde\Theta_3+2\tau^5n_1^5\tilde\Theta_5+2\tau^7n_1^7\tilde\Theta_7+2\tau^9n_1^9\tilde\Theta_9. \qquad \text{(C.2)}$$

Остальные слагаемые:

$$N_{YXX}\Xi_{\Omega_0\Omega_1\Omega_1}=N_{YXX}\left[\left[\tau A_{1,0}\Theta_1+\tau^5A_{5,0}\tilde\Theta_5+\tau^7A_{7,0}\tilde\Theta_7,\tau n_1\Theta_1+\tau^5n_1^5\tilde\Theta_5+\tau^7n_1^7\tilde\Theta_7\right],\tau n_1\Theta_1\right]=$$
$$=\tau^7N_{YXX}\left\{A_{1,0}n_1^6\Xi_{\Theta_1\Theta_5\Theta_1}+A_{5,0}n_1^2\Xi_{\Theta_5\Theta_1\Theta_1}\right\}+\tau^9N_{YXX}\left(A_{1,0}n_1^8\Xi_{\Theta_1\Theta_7\Theta_1}+A_{7,0}n_1^2\Xi_{\Theta_7\Theta_1\Theta_1}\right), \qquad \text{(C.3.1)}$$

$$N_{YXY}\Xi_{\Omega_0\Omega_1\Omega_0}=N_{YXY}\left[\left[\tau A_{1,0}\Theta_1+\tau^5A_{5,0}\tilde\Theta_5+\tau^7A_{7,0}\tilde\Theta_7,\tau n_1\Theta_1+\tau^5n_1^5\tilde\Theta_5+\tau^7n_1^7\tilde\Theta_7\right],\tau A_{1,0}\Theta_1\right]=$$
$$=\tau^7N_{YXY}\left\{\left(A_{1,0}\right)^2n_1^5\Xi_{\Theta_1\tilde\Theta_5\Theta_1}+A_{1,0}A_{5,0}n_1\Xi_{\tilde\Theta_5\Theta_1\Theta_1}\right\}+\tau^9N_{YXY}\left(\left(A_{1,0}\right)^2n_1^7\Xi_{\Theta_1\tilde\Theta_7\Theta_1}+A_{1,0}A_{7,0}n_1\Xi_{\tilde\Theta_7\Theta_1\Theta_1}\right), \qquad \text{(C.3.2)}$$

$$N_{YXXXX}\Xi_{\Omega_0\Omega_1\Omega_1\Omega_1\Omega_1}=\tau^9N_{YXXXX}\left(A_{1,0}n_1^8\Xi_{\Theta_1\tilde\Theta_5\Theta_1\Theta_1\Theta_1}+A_{5,0}n_1^4\Xi_{\tilde\Theta_5\Theta_1\Theta_1\Theta_1\Theta_1}\right), \qquad \text{(C.3.3)}$$

$$N_{YXXXY}\Xi_{\Omega_0\Omega_1\Omega_1\Omega_1\Omega_0}=\tau^9N_{YXXXY}\left(\left(A_{1,0}\right)^2n_1^7\Xi_{\Theta_1\tilde\Theta_5\Theta_1\Theta_1\Theta_1}+A_{1,0}A_{5,0}n_1^3\Xi_{\tilde\Theta_5\Theta_1\Theta_1\Theta_1\Theta_1}\right), \qquad \text{(C.3.4)}$$

$$N_{YXXYY}\Xi_{\Omega_0\Omega_1\Omega_1\Omega_0\Omega_0}=\tau^9N_{YXXYY}\left(\left(A_{1,0}\right)^3n_1^6\Xi_{\Theta_1\tilde\Theta_5\Theta_1\Theta_1\Theta_1}+\left(A_{1,0}\right)^2A_{5,0}n_1^2\Xi_{\tilde\Theta_5\Theta_1\Theta_1\Theta_1\Theta_1}\right), \qquad \text{(C.3.5)}$$

$$N_{YXYYY}\Xi_{\Omega_0\Omega_1\Omega_0\Omega_0\Omega_0} = \tau^9 N_{YXYYY}\left(\left(A_{1,0}\right)^4 n_1^5 \Xi_{\Theta_1\tilde{\Theta}_5\Theta_1\Theta_1\Theta_1} + \left(A_{1,0}\right)^3 A_{5,0} n_1 \Xi_{\tilde{\Theta}_5\Theta_1\Theta_1\Theta_1\Theta_1}\right). \quad (C.3.6)$$

Итоговое выражение для $\Omega^{(1)}$:

$$\begin{aligned}\Omega^{(1)} &= \tau A_{1,1}\Xi_{\Theta_1} + \tau^5 A_{5,1}\Xi_{\tilde{\Theta}_5} + \tau^7\left(A_{7,1}\Xi_{\tilde{\Theta}_7} + B_{7,1}\Xi_{\tilde{\Theta}_5\Theta_1\Theta_1}\right) \\ &+ \tau^9\left(A_{9,1}\Xi_{\tilde{\Theta}_9} + B_{9,1}\Xi_{\tilde{\Theta}_7\Theta_1\Theta_1} + C_{9,1}\Xi_{\tilde{\Theta}_5\Theta_1\Theta_1\Theta_1\Theta_1}\right).\end{aligned} \quad (C.4)$$

Здесь

$$A_{1,1} = n_0 + 2n_1 \equiv A_{1,0} + 2n_1, \quad (C.4.1)$$

$$A_{3,1} = n_0^3 + 2n_1^3 \equiv A_{3,0} + 2n_1^3, \quad (C.4.2)$$

$$A_{5,1} = n_0^5 + 2n_1^5 \equiv A_{5,0} + 2n_1^5, \quad (C.4.3)$$

$$A_{7,1} = n_0^7 + 2n_1^7 \equiv A_{7,0} + 2n_1^7, \quad (C.4.4)$$

$$A_{9,1} = n_0^9 + 2n_1^9 \equiv A_{9,0} + 2n_1^9, \quad (C.4.5)$$

$$B_{7,1} = B_{7,0} + N_{YXY}\left\{A_{1,0}A_{5,0}n_1 - \left(A_{1,0}\right)^2 n_1^5\right\} + N_{YXX}\left\{A_{5,0}n_1^2 - A_{1,0}n_1^6\right\}, \quad (C.4.6)$$

$$B_{9,1} = B_{9,0} + N_{YXX}\left\{A_{7,0}n_1^2 - A_{1,0}n_1^8\right\} + N_{YXY}\left\{A_{1,0}A_{7,0}n_1 - \left(A_{1,0}\right)^2 n_1^7\right\}, \quad (C.4.7)$$

$$\begin{aligned}C_{9,1} &= C_{9,0} + N_{YXXXX}\left\{A_{5,0}n_1^4 - A_{1,0}n_1^8\right\} + N_{YXXXY}\left\{A_{1,0}A_{5,0}n_1^3 - \left(A_{1,0}\right)^2 n_1^7\right\} \\ &+ N_{YXXYY}\left\{\left(A_{1,0}\right)^2 A_{5,0}n_1^2 - \left(A_{1,0}\right)^3 n_1^6\right\} + N_{YXYYY}\left\{\left(A_{1,0}\right)^3 A_{5,0}n_1 - \left(A_{1,0}\right)^4 n_1^5\right\},\end{aligned} \quad (C.4.8)$$

Вычисление выражения $\Omega^{(2)}$ показывает, что только одна функция, C_9, будет иметь слагаемые, отличающиеся по форме от предыдущего шага.

В итоге

$$\begin{aligned}\Omega^{(m)} &= \tau A_{1,m}\Xi_{\Theta_1} + \tau^5 A_{5,m}\Xi_{\tilde{\Theta}_5} + \tau^7\left(A_{7,m}\Xi_{\tilde{\Theta}_7} + B_{7,m}\Xi_{\tilde{\Theta}_5\Theta_1\Theta_1}\right) + \\ &+ \tau^9\left(A_{9,m}\Xi_{\tilde{\Theta}_9} + B_{9,m}\Xi_{\tilde{\Theta}_7\Theta_1\Theta_1} + C_{9,m}\Xi_{\tilde{\Theta}_5\Theta_1\Theta_1\Theta_1\Theta_1}\right).\end{aligned} \quad (C.5)$$

Здесь

$$A_{1,m} = A_{1,m-1} + 2n_m, \quad (C.6.1)$$

$$A_{5,m} = A_{5,m-1} + 2n_m^5, \quad (C.6.2)$$

$$A_{7,m} = A_{7,m-1} + 2n_m^7, \quad (C.6.3)$$

$$A_{9,m} = A_{9,m-1} + 2n_m^9, \quad (C.6.4)$$

$$B_{7,m} = B_{7,m-1} + N_{YXY}\left\{A_{1,m-1}A_{5,m-1}n_m - \left(A_{1,m-1}\right)^2 n_m^5\right\} + N_{YXX}\left\{A_{5,m-1}n_m^2 - A_{1,m-1}n_m^6\right\}, \quad (C.6.5)$$

$$B_{9,m} = B_{9,m-1} + N_{YXX}\left\{A_{7,m-1}n_m^2 - A_{1,m-1}n_m^8\right\} + N_{YXY}\left\{A_{1,m-1}A_{7,m-1}n_m - \left(A_{1,m-1}\right)^2 n_m^7\right\}, \quad (C.6.6)$$

$$C_{9,m} = C_{9,m-1} + N_{YXXXX}\left\{A_{5,m-1}n_m^4 - A_{1,m-1}n_m^8\right\} + N_{YXXXY}\left\{A_{1,m-1}A_{5,m-1}n_m^3 - \left(A_{1,m-1}\right)^2 n_m^7\right\} +$$
$$+N_{YXXYY}\left\{\left(A_{1,m-1}\right)^2 A_{5,m-1}n_m^2 - \left(A_{1,m-1}\right)^3 n_m^6\right\} + N_{YXYYY}\left\{\left(A_{1,m-1}\right)^3 A_{5,m-1}n_m - \left(A_{1,m-1}\right)^4 n_m^5\right\} + \quad \text{(C.6.7)}$$
$$+N_{YXX}B_{7,m-1}n_m^2 + N_{YXY}B_{7,m-1}A_{1,m-1}n_m.$$

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