ADEQUACY CRITERIA FOR NON-RADIATIVE LIMIT IN NUMERICAL STUDIES OF DARK PLASMA

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Abstract. The characteristic features of Darwin's (non-radiative) description of electromagnetic fields are considered. Criteria for the physical adequacy of the formalism are obtained within the framework of a self-consistent approach to the model representation of nonlinear processes of rarefied plasma. The implementation of the proposed estimates of the adequacy of the non-radiative approximation based on numerical researches of the Weibel instability.

Keywords: *rarefied plasma, self-consistent approach, non-radiative limit, Lagrangian interaction, Weibel instability*

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INTRODUCTION

As is known, the self-consistent approach [1], which takes into account the interdependent influence of the phase distribution of particles and internal electromagnetic fields, is very effective in the study of nonequilibrium states of hot plasma, which caused its active use in many fundamental and applied problems of nonlinear plasma physics. Mathematically, it was represented by a system of the Vlasov equations describing in detail the evolution of each of the components of the collisionless plasma and the Maxwell equations, which most fully reflect the dynamics of the internal electromagnetic fields.

At the same time for low-frequency weakly relativistic systems free electromagnetic fields of radiative nature play a minor role since they are small in comparison with self-consistent ones and are characterized by essentially smaller space-time scales. Obviously, the numerical analysis of such systems in the framework of Maxwell's formalism turns out to be excessively detailed and, as a consequence, rather expensive.

In this connection, it is reasonable to refer to the reduced field descriptions, of which the Darwin approximation (radiationless limit) [2] seems to be the most interesting, since it excludes free electromagnetic waves from consideration. The non-triviality of the approximation consists in the fact that, neglecting the delay, it preserves a number of induction effects, in particular, those related to Faraday's law.

The mentioned properties of the Vlasov-Darvin model allow us to designate its subject area: weakly relativistic and relatively low-frequency phenomena of a rarefied magnetically active plasma caused by collective interactions of particles.

At the same time, we would like to have clearer guidelines for the correct physical application of the formalism under consideration, if possible, in the form of certain criteria. For these purposes, let us consider the analytical representation and characteristic features of the Darwin approximation of electromagnetic fields.

DARWIN'S APPROACH

In the SGCE system, the equations of the Darwinian representation of the electromagnetic field have the following form

$$\nabla \vec{E} = 4\pi\rho \ , \tag{1}$$

$$\nabla \times \vec{E} = - \ \frac{1}{c}\frac{\partial \vec{B}}{\partial t} \ , \tag{2}$$

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}_p}{\partial t} \, , \tag{3}$$

$$\nabla \vec{B} = 0 \, , \tag{4}$$

$$\vec{E} = \vec{E}_p + \vec{E}_v : \quad \nabla \times \vec{E}_p = 0 \, , \ \ \nabla \vec{E}_v = 0 \, . \tag{5}$$

Here$\vec{E}_p, \vec{E}_v$ are the potential (longitudinal) and vortex (transverse) components of the electric field, respectively.

Thus the charge continuity equation, which must be identically satisfied, can be written in a modified form:

$$\nabla\vec{J} + \frac{\partial \rho}{\partial t} = \nabla \vec{J} + \frac{\partial}{\partial t}\left(\frac{1}{4\pi}(\nabla\vec{E}_p)\right). \tag{6}$$

Let us note the characteristic points of the formalism under consideration.

First of all, as it is known, the Helmholtz decomposition, actively used by the Darwinian representation, is ambiguous and requires additional conditions for its certainty. This question is considered in detail in [3] and within the framework of the present paper is not of special interest.

The main goal, set by Darwin, was to obtain as accurate as possible Lagrangian function for the system of particles and fields with instantaneous long-range interaction. As it turned out, at comparatively small ratio of the thermal speed(v) to the speed of light(c) such system can be described through non-delayed potentials by the Lagrangian of interaction accurate to values of the order of$(v/c)^2$. Therefore, structurally answering to the notion of weak relativity, it does not include the braking radiation of charges, which is represented by terms not lower than the third order of smallness in the parameter expansion $(v/c) << 1$ of the full Lagrangian function [4].

Thus, following the physical content of the Darwinian approximation, the system of equations (1) - (5) should not describe a free electromagnetic wave, i.e. electric and magnetic fields should be exclusively self-consistent.

Let us show it by applying the operationrot to equation (3). Then, using the known transformation of the double rotor product, we obtain

$$\Delta\vec{B} = -\frac{4\pi}{c}(\nabla \times \vec{J}). \tag{7}$$

As can be easily seen from equations (1), (2) and (7) (taking into account the expansion (5)), the internal electromagnetic field is really determined only by the current values of the charge and current densities. The latter, we note in passing, determines the numerical economy of the field calculation.

Finally, we shall convince that the collective interactions of the sparse weakly relativistic $(T << mc^2)$ plasma, corresponding to the physical filling of the self-consistent approach, the Darwinian formalism describes very precisely.

For this purpose, we consider the instantaneous long-range effect on characteristic plasma scales (,$r_{De}\omega_{pe}$), where the lag time can be estimated as

$$\tau_3 \simeq \frac{r_{De}}{c} \ll \frac{r_{De}}{v_T} \simeq \omega_{pe}^{-1}. \tag{8}$$

Consequently, the distribution of particles moving with velocities$\sim v_T$, does not have time to change appreciably at timesτ_3 . That is with respect to the instantaneous values of the self-consistent fields the corrections due to the delay are vanishingly small corrections.

In conclusion of a brief review of the main features of the radiationless limit, we note that it differs externally from the full electromagnetic description only by the omitted transverse component of the bias current. This point is very important in the context of the present work, since it seems natural to search for a condition of correctness of the Darwin formalism in problems of kinetics of a rarefied plasma by evaluating the possible influence of the omitted part of the bias current on the development of a particular plasma process.

ADEQUACY CRITERIA

The required estimate can be obtained by considering the contributions of the omitted and remaining parts of the bias current to the generation of the internal electromagnetic field. For this purpose, we will use the dimensional (more precisely, scaling) analysis, within the framework of which we will assume that significant changes in the basic values of the Darwinian system of equations occur on characteristic scales $(L, T\ )$ of some hypothetical process. In this case, the averaged values of$|\nabla \times \vec{u}|$ or$|\nabla \vec{u}|$ can be estimated as(u/L) , and the averaged value of$|\ \partial \vec{u}/\ \partial t|$ as .(u/T)

Then it follows from equation (2):

$$|\nabla \times \vec{E}| = |\nabla \times \vec{E}_v| \sim E_v/L\ ; \quad |\ \partial \vec{B}/\ \partial t| \sim B/T. \tag{9}$$

And the assessment is correct.

$$E_v \sim (BL)/(cT). \tag{10}$$

Similarly, taking into account equation (6), we can write:

$$|\nabla \vec{j}| \sim J/L;\ \left|\frac{\partial}{\partial t}\left(\frac{1}{4\pi}(\nabla \vec{E}_p)\right)\right| = \left|\nabla \frac{1}{4\pi}\frac{\partial \vec{E}_p}{\partial t}\right| \sim |\ \partial \vec{E}_p/\ \partial t|/(4\pi L). \tag{11}$$

Whence it follows that

$$|\ \partial \vec{E}_p/\ \partial t| \sim 4\pi J. \tag{12}$$

Next, from equation (3) we obtain, taking into account the estimation (12), the relations:

$$\begin{aligned} |\nabla \times \vec{B}| \sim B/L;\ &\left|\frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{E}_p}{\partial t}\right| \\ &\leq \left|\frac{4\pi}{c}\vec{j}\right| + \left|\frac{1}{c}\frac{\partial \vec{E}_p}{\partial t}\right| \sim \left(\frac{4\pi}{c}J + \frac{4\pi}{c}J\right). \end{aligned} \tag{13}$$

And we use them to find the boundary values of$|\vec{B}|$:

$$B \leq \frac{4\pi}{c} JL. \tag{14}$$

Finally, using the estimate (10) and the upper bound$|\vec{B}|$, we obtain:

$$|\,\partial \vec{E}_v / \,\partial t| \sim E_v / T \sim (BL)/(cT^2) \leq 4\pi J \left(\frac{L}{cT}\right)^2. \tag{15}$$

Thus, taking into account the expressions (12) and (15), the ratio of the values of transverse and longitudinal components of the bias current has the estimation

$$\frac{|\,\partial \vec{E}_v / \,\partial t|}{|\,\partial \vec{E}_p / \,\partial t|} \sim \left(\frac{L}{cT}\right)^2. \tag{16}$$

This estimate, in fact, determines the criterion of physical adequacy of the Darwin approximation of fields in problems of kinetics of rarefied plasma:

$$\left(\frac{L}{cT}\right)^2 << 1. \tag{17}$$

Indeed, in this case, the contribution of the transverse component of the bias current to the development of the investigated process against the contribution of its longitudinal component is vanishingly small and the results obtained within the framework of the Vlasov-Darwin model are physically reliable.

We emphasize that the analysis of relation (16) is correct only if the inequality is unconditionally satisfied

$$\frac{v_T}{c} << 1, \tag{18}$$

which in the present context can be called a condition (criterion) of weak relativism. This criterion of adequacy follows from initial physical preconditions of construction of the Darwinian Lagrangian of interaction, but has a relation to the whole self-consistent formalism.

Note that the following, so to speak, wave interpretation of the found criterion (17) is possible.

Let the linear size of the model domain (l_{sys}) be of the order of the spatial scale (L) of the process, and let its characteristic time (T) determine some, let us call it characteristic, frequency .ω_{ch}

Then the expression($L/(cT)$) can be interpreted as the ratio of the length of the modeling region to the wavelength (λ) with frequency (ω_{ch}) in vacuum:

$$\left(\frac{L}{cT}\right) \sim \left(\frac{l_{sys}}{\lambda_{ch}}\right) \tag{19}$$

Hence, the use of the radiation-free limit within the self-consistent approach is correct if the linear size of the model system is essentially smaller than the wavelength with characteristic frequency ($\simeq 2\pi/T$) in vacuum.

Such an interpretation of the obtained criterion is especially clear in the case of numerical analysis of various kinds of instabilities. In this connection, let us consider, for example, the justification of the correctness of the Vlasov-Darwin model in a real numerical study of the Weibel instability [5].

WAIBEL INSTABILITY

As is known, anisotropic distribution of electrons by velocities in a homogeneous rarefied plasma can cause spontaneous occurrence and rapid amplification of the magnetic field transverse to the accentuated component of particle velocity - the Weibel electromagnetic instability (WE) [6]. Having a certain specificity of manifestations in a number of areas of plasma physics [7], it is characterized by the obligatory presence of an essentially nonlinear mode, which determines the spatial and temporal scales of the phenomenon ( ).L, T

In this case, the well-developed linear theory of electromagnetic instabilities [8] allows us to make quantitative estimates of the characteristic parameters of VN. In particular, to find the dependences of the incrementγ on the wave number$k$ (at different values of the initial anisotropy index of the medium$A$) from the dispersion equation obtained in the cited paper

$$k_x^2 c^2 - \omega^2 = \omega_{pe}^2 \left(A + (A+1) \frac{\omega}{k_x u_x} Z\left(\frac{\omega}{k_x u_x}\right)\right). \qquad (20)$$

Here$A = (u_z^2/u_x^2 - 1)$ (anisotropic electron distribution is characterized by the ratio $u_z > u_x = u_y$ of the thermal velocity components), $\omega_{pe} = \sqrt{4\pi n_0 q^2/m}$ is the Langmuir frequency, and$Z$ is the so-called dispersion function [5].

The unstable roots of equation (20) define standing waves with imaginary positive valueω , conditioning the development of HV. They are in the range from$k_x c/\omega_{pe} = 0$ to$k_x c/\omega_{pe} = \sqrt{A}$ for any$A > 0$. At the same time, the instability increment$\gamma = -i\omega$ has one maximum in this region, which allows us to identify the most active mode (see Fig.

1).

In [5], a uniform spatial distribution of electrons and single-charged ions, which had the appearance of a positive background due to inertness, was specified. The components of the thermal velocity of electrons were assumed to be , $u_z = 0.1\ [c] u_x = u_y = 0.0316\ [c]$, which determined the initial anisotropy index .$A = 9$

Numerical solution of equation (20) (see, e.g., [9]) for the adopted values of the basic parameters gave the values $\gamma_{\max} \approx 0.037\ [\omega_{pe}]$ and $k_{\max} \approx 1.2\ [\omega_{pe}/c]$, which corresponded to the wavelength .$\lambda_{\max} \approx 5.2\ [c/\omega_{pe}]$

The obtained values of the wavelength $\lambda_{\max}$ and increment $\gamma_{\max}$ of the most active mode, in fact, determining the development of the Weibel instability, allow us to estimate the characteristic spatial and temporal scales of the process:

$$T \sim 1/\gamma_{\max}, \qquad L \sim \lambda_{\max} \tag{21}$$

and taking into account the value of the accentuated component of thermal velocity to find the values of parameters of weak relativity of the plasma system

$$(v_T/c) < (u_z/c) = 0.1 \tag{22}$$

and correctness of the Darwinian approximation of electromagnetic fields

$$(L/(cT))^2 \sim (\lambda_{\max} \cdot \gamma_{\max})^2 \approx 0.04\,. \tag{23}$$

Let's notice, that the ratio of linear size of model system l_{sys} and wavelength with frequency ω_{ch} in vacuum has (in units of c/ω_{pe}) the form

$$l_{sys} \sim \lambda_{\max} \approx 5.2 << \lambda_{ch} \sim cT \approx 27. \tag{24}$$

Thus, based on the obtained estimates, it can be stated that the results of numerical analysis of the Weibel instability within the framework of the self-consistent approach with the radiation-free (Darwin) approximation of the internal electromagnetic fields will be physically reliable.

CONCLUSION

The proposed approach allows us to obtain a priori information about the possibility of using the Darwinian representation of electromagnetic fields within the framework of the self-consistent formalism at the stage of analytical prediction of the main parameters of the studied phenomenon of a sparse magnetically active plasma. Taking into account the significant numerical economy of the radiation-free limit in comparison with the full (Maxwellian) description, it seems reasonable to include the considered technique in the general formulation of computer experiments, especially in the study of large-scale plasma processes.

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FIGURE CAPTION

Figure 1. Increments of the Weibel instability for different A.

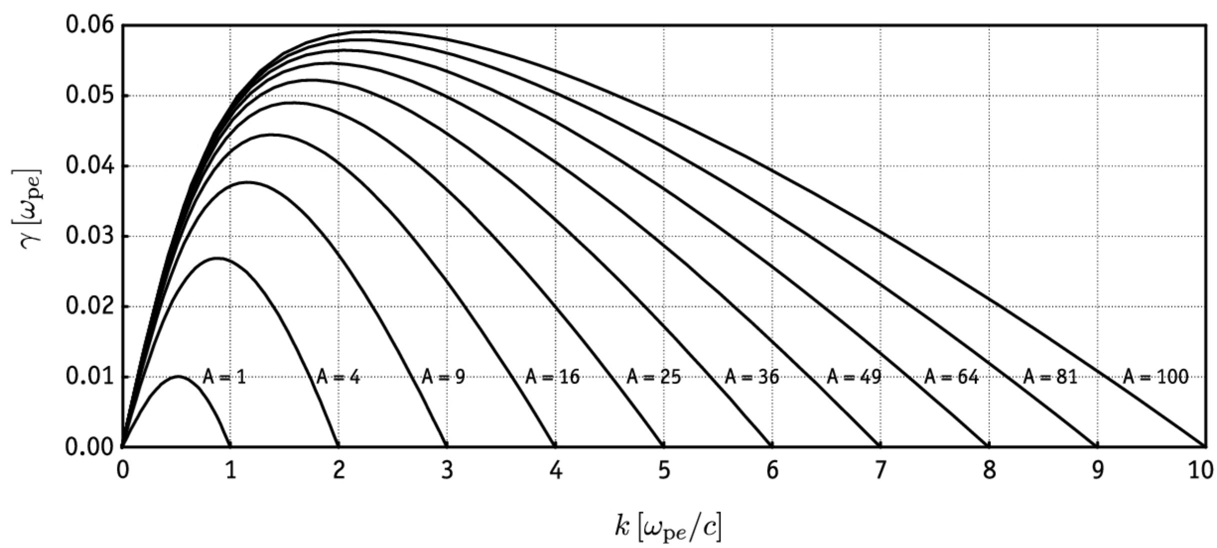

Fig. 1.