FEATURES OF ACOUSTIC WAVE PROPAGATION IN NARROW PIPES OF VARIABLE CROSS-SECTION, CONSIDERING THE ATTACHED MASS

© 2025 V. A. Gusev*, K. O. Komarovskii

Lomonosov Moscow State University, Moscow, Russia

**e-mail: vgusev@bk.ru*

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Abstract. The phenomena that occur during the propagation of acoustic waves in narrow tubes with variable cross-section are studied. A special law of cross-sectional variation has been defined for the tube. The mode of wave tunneling through narrowing is investigated. The influence of the attached mass on the boundaries of the narrowing area of the tube is taken into account. The frequency dependences of the wave are constructed taking into account the attached mass.

Keywords: *attached mass, wave tunneling, Webster equation, transmission coefficient*

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INTRODUCTION

Pipes with constant and variable cross-section are an integral part of all sound conduits used in practice. For this reason, the study of the laws of sound propagation in such a system is of great importance for solving all acoustic questions related to experimental studies and processing of their results.

A great number of works have been devoted to the study of sound wave propagation in an infinite acoustic waveguide. A special place among the problems under consideration is occupied by problems in which the waveguide has a changing cross-sectional profile. This class of problems is of great interest because in the zone of sectional change the wave front is reconstructed, which, in turn, generates a number of interesting effects.

This work is devoted to analyzing the peculiarities of acoustic wave propagation in composite tubes of variable cross-section, in which different sections may have different acoustic characteristics. At the junction of such sections, the attached mass, introduced to describe the transient process of wave transformation, plays a significant role. To describe these effects, it is necessary to solve the full problem of wave passage and reflection through the constriction region. In addition, a brief review of the basic equations for describing acoustic waves, taking into account the nonlinearity and dissipation of the medium, is given in the paper.

SOUND PROPAGATION THROUGH A TUBE OF VARIABLE CROSS-SECTION

Consider a situation when an acoustic wave propagates in a narrow tube, one of the sections of which has a smooth constriction with a variable cross-section (see Fig. 1). The medium parameters - densityρ_1 and sound velocityc_1 - in region II with narrowing ($0 < x < d$) generally differ from the medium parameters - densityρ_0 and sound velocityc_0 - in sections I and III with constant cross-section. Consideration of identical media in sections I and III simplifies the final expressions, but does not limit the generality of the obtained results.

In the linear approximation, waves in horns, tubes, concentrators, and other waveguide systems with variable cross section$S(x\ )$ are described by Webster's equation [1-3]:

$$\frac{1}{S(x)}\frac{\partial}{\partial x}\left(S(x)\frac{\partial p}{\partial x}\right)-\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}=0. \qquad (1)$$

Here$x$ is the coordinate counted along the system axis,p is the acoustic pressure,c is the speed of sound, hereafterρ is the density of the medium.

Webster's equation (1) is applicable to tubes whose characteristic radius is small compared to the wavelength:$r_0 \ll \lambda$. The basic approximation made in deriving this equation is that the wave should maintain one-dimensional propagation despite changes in the cross-section of the tube. This condition is well satisfied for narrow tubes in which only the piston mode is effectively excited and propagates due to the presence of dispersion. For tubes with a large characteristic cross-sectional radius, this equation will also apply if the radius changes relatively slowly as propagation proceeds. If the cross-sectional area changes rapidly enough in some regions of the tube, the planar character of the motion in the transition region may be distorted. These distortions of the character of motion are usually phenomenologically accounted for by the introduction of an attached mass describing the inertial drag [4,5]. The conditions of applicability of Webster's equation to the problem solved in this paper will also be discussed below.

PASSAGE OF AN INTENSE WAVE THROUGH A TUBE SECTION WITH CONSTRICTION TAKING INTO ACCOUNT LOW-FREQUENCY DISPERSION AND ATTACHED MASS

Even in the case of the simplest, linear Webster equation (1), the study reveals very curious effects. One of them is the tunneling of acoustic waves as they propagate through a smoothly narrowing waveguide channel [6-10]. This mode corresponds to the passage through the narrowing of an inhomogeneous wave and was considered in [6, 7].

As mentioned above, in the region of relatively abrupt changes in cross-sectional area, vortex motions may occur, for accounting for which an attached mass is introduced. Since the cross-sectional profile, presented in Fig. 1, although continuous, contains a discontinuity of the derivative, the appearance of the attached mass is also possible here. This factor was not considered in [6, 7]. This work is devoted to the study of the influence of the attached mass on the tunneling effect at a smooth narrowing of the tube cross section.

To obtain exact solutions of equation (1), we introduce instead of the pressure$p$ a new function :F

$$p(x,t) = \frac{F(x,t)}{\sqrt{S(x)}}. \tag{2}$$

For this function, equation (1) will take the form:

$$\frac{\partial^2 F}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 F}{\partial t^2} = \frac{1}{\sqrt{S(x)}}\frac{\partial^2 \sqrt{S(x)}}{\partial x^2} F. \tag{3}$$

Let us set the cross-sectional area in equation (3) in a special form

$$S(x) = S_m \operatorname{sh}^2\left[\theta\left(x - \frac{d}{2}\right)\right], \theta = \frac{2}{d}\operatorname{arch}\frac{1}{\sqrt{S_m}}. \quad (4)$$

Here, the constantsS_m, d are the geometric characteristics of the variable thickness section:S_m is the minimum dimensionless area of contraction achieved at the coordinate value$x = \frac{d}{2}$, and$d$ is the length of this section.

Then at we reduce equation (3) to the form:

$$\frac{\partial^2 F}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 F}{\partial t^2} = \theta^2 F. \quad (5)$$

Thus, the original equation was reduced to an equation with constant coefficients, which allows us to obtain an exact solution. Compared to the wave equation, equation (5) contains an additional summand, which is responsible for the low-frequency dispersion. This summand allows choosing different modes of wave passage depending on the frequency.

Particularly interesting is the tunneling mode of the wave. Speaking about tunneling, we often mean quantum effects of penetration of particles through potential barriers, the height of which is greater than the energy of the particles themselves. In this case we are talking about the situation when the inhomogeneous wave mode is realized in the constriction region. In this case, the attenuation in the low-frequency range is insignificant, and there is no phase shift [6]. As a result, a plateau is formed on the frequency dependence of the transmission coefficient, where its value is close to unity. The wave passes practically without losses and distortions.

By substituting a harmonic wave of frequency ,$\omega F = F_0 e^{-i\omega t}$, equation (5) can be reduced to the form:

$$\frac{\partial^2 F}{\partial x^2} + (k_1^2 - \theta^2)F = 0, \tag{6}$$

where$k_1 = \frac{\omega}{c_1}$ is the wave number in the medium.

Obviously, the tunneling effect is possible only in some narrow range of relevant parameters, which is yet to be found. The wave tunneling mode corresponds to the value$\theta^2 > k_1^2$. This mode can be realized for low frequencies$k_1 < \theta$ or

$$\omega < \frac{2c}{d} \operatorname{arch} \frac{1}{\sqrt{S_m}}, k_1 d < 2 \operatorname{arch} \frac{1}{\sqrt{S_m}}. \tag{7}$$

As can be seen, the tunneling mode really corresponds to the situation when the contraction length *d is* smaller than the wavelength. Thus, the change of the cross-sectional area is quite abrupt and the question arises about the legitimacy of using the model of Webster's equation (1) in this case. The considered situation with a smooth but rather fast change of the cross-sectional area is an intermediate variant between two model situations: 1) smooth and slow narrowing, for which Webster's equation is an adequate model, and 2) abrupt jumping narrowing, which is modeled by a thin baffle and for which full penetration is possible only at zero frequency. The second of these models completely omits the effect of the smoothness of the change in cross-sectional area. The adequacy of the application of the Webster equation model for this case can be justified by the existence of a limiting transition from one model to the other. Indeed, by taking the length of the constriction to zero, we see that the constriction will jump and the complete passage of sound will only be at zero frequency. The effects arising at a finite length of the constriction will be modeled, as usual, by adding an attached mass.

To find the reflection and transmission coefficients of the wave, let us write down the boundary conditions in the region:$0 < x < d$. We require continuity of the pressure$p$ and velocity$u$:

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x}, u = -\frac{i}{k\rho c}\frac{dp}{dx}.$$

In the places of contact of tubes of different cross-section, generally speaking, the character of the medium motion changes, the wave ceases to be flat. For phenomenological description of the transition process, an attached mass is introduced. It can be expected that in the case of a smooth change in the cross-section of the tubes this mass is smaller than for the case of an abrupt jump in cross-section. Nevertheless, it is important to see how significant the influence of the attached mass is.

After substituting the inhomogeneous wave solution into the boundary conditions, we obtain a system of equations:

$$\begin{cases} P_i + P_r = P_+ + P_- + iP'(0), \\ P_i - P_r = i\dfrac{\alpha}{k_1}[P_+(\mu - b) - P_-(\mu + b)], \\ P_+e^{-\mu d} + P_-e^{\mu d} = P_t + iP'(d), \\ i\dfrac{\alpha}{k_1}[P_+e^{-\mu d}\Big((\mu + b) - P_-e^{\mu d}(\mu - b)\Big] = P_t. \end{cases} \quad (8)$$

In formula (8)P_i, P_r, P_t - amplitudes of incident, reflected, passed waves,P_+, P_- - amplitudes of waves propagating inside the tube - in region II, , ,$P'(0) = \frac{M_1 \omega u_{\mathrm{I}}}{S_1} P'(d) = \frac{M_2 \omega u_{\mathrm{III}}}{S_2} M_1, M_2$ - attached mass at the tube inlet and outlet, respectively, $u_{\mathrm{I}}, u_{\mathrm{III}}$ - oscillatory velocity at the boundary ,$x = 0 x = d$, respectively, , ,$b \equiv \frac{2}{d}\sqrt{1 - S_m}\mathrm{arch}\frac{1}{\sqrt{S_m}} \mu = \sqrt{\theta^2 - k_1^2} \alpha = \frac{\rho_0 c_0}{\rho_1 c_1}$ - ratio of acoustic impedances of the two media,S_1, S_2 - dimensionless cross-sectional areas of the tube at$x = 0$ and$x = d$.

Solving the system (8), we find the expressions forP_+ and :P_-

$$P_+ = \frac{2P_i}{\Delta} e^{\overline{\mu}} \left[1 - \gamma \frac{M_1 \omega}{S_1} \frac{1}{\rho_0 c_0} \frac{\alpha}{\overline{\overline{k_1}}} (\overline{\mu} - \overline{b}) - i \frac{\alpha}{\overline{\overline{k_1}}} (\overline{\mu} - \overline{b})\right], \tag{9}$$

$$P_- = -\frac{2P_i}{\Delta} e^{-\overline{\mu}} \left[1 + \gamma \frac{M_1 \omega}{S_2} \frac{1}{\rho_0 c_0} \frac{\alpha}{\overline{\overline{k_1}}} (\overline{\mu} + \overline{b}) - i \frac{\alpha}{\overline{\overline{k_1}}} (\overline{\mu} + \overline{b})\right], \tag{10}$$

where

$$\Delta = e^{\overline{\mu}} \left[1 - \frac{M_1 \omega}{S_1} \frac{1}{\rho_0 c_0} \frac{\alpha}{\overline{\overline{k_1}}} (\overline{\mu} - \overline{b}) + i \frac{\alpha}{\overline{\overline{k_1}}} (\overline{\mu} - \overline{b})\right] \left[1 - \gamma \frac{M_1 \omega}{S_2} \frac{1}{\rho_0 c_0} \frac{\alpha}{\overline{\overline{k_1}}} (\overline{\mu} - \overline{b}) + \right.$$

$$\left. i \frac{\alpha}{\overline{\overline{k_1}}} (\overline{\mu} - \overline{b})\right] - e^{-\overline{\mu}} \left[1 + \frac{M_1 \omega}{S_1} \frac{1}{\rho_0 c_0} \frac{\alpha}{\overline{\overline{k_1}}} (\overline{\mu} + \overline{b}) - i \frac{\alpha}{\overline{\overline{k_1}}} (\overline{\mu} + \overline{b})\right] \left[1 + \gamma \frac{M_1 \omega}{S_2} \frac{1}{\rho_0 c_0} \frac{\alpha}{\overline{\overline{k_1}}} (\overline{\mu} + \right.$$

$$\left. \overline{b}) - i \frac{\alpha}{\overline{\overline{k_1}}} (\overline{\mu} + \overline{b})\right],$$

and dimensionless parameters are introduced:

$$\overline{k_1} = k_1 d; \; ; \; .\overline{\mu} = \mu d \overline{b} = b d \gamma = \frac{M_2}{M_1}$$

From equations (8) to (10) we find the wave travel coefficient :T

$$T = \frac{P_t}{P_i} = 2i \frac{\alpha}{\overline{\overline{k_1}}} \frac{1}{\Delta} \left\{ \left[(\overline{\mu} + \overline{b}) \left(\left(1 - \gamma \frac{M_1 \omega}{S} \frac{1}{\rho_0 c_0} \frac{\alpha}{\overline{\overline{k_1}}} (\overline{\mu} - \overline{b}) + i \frac{\alpha}{\overline{\overline{k_1}}} (\overline{\mu} - \overline{b}) \right) \right) \right] + \right.$$

$$\left. + \left[(\overline{\mu} - \overline{b}) \left(\left(1 + \gamma \frac{M_1 \omega}{S} \frac{1}{\rho_0 c_0} \frac{\alpha}{\overline{\overline{k_1}}} (\overline{\mu} + \overline{b}) - i \frac{\alpha}{\overline{\overline{k_1}}} (\overline{\mu} + \overline{b}) \right) \right) \right] \right\}. \tag{11}$$

Here, for simplification of the notation,$S_1 = S_2 = S$ is used. If we put$M_1 = 0$, which means the absence of the attached mass, the expression (11) will take the form:

$$T = \frac{\frac{i 4 \alpha \overline{\mu}}{\overline{\overline{k_1}}}}{e^{\overline{\mu}} \left[1 + i \frac{\alpha}{\overline{\overline{k_1}}} (\overline{\mu} - \overline{b})\right]^2 - e^{-\overline{\mu}} \left[1 - i \frac{\alpha}{\overline{\overline{k_1}}} (\overline{\mu} + \overline{b})\right]^2}. \tag{12}$$

We plot the frequency dependences of the square of the wave transmission coefficient for the ratio of the parameters$\alpha = 2$ and three values of the minimum tube cross-section$S_m = 0.09, 0.34, 0.71$ (curves 1, 2, 3 in Figs. 2, 3, respectively).

In order to emphasize the influence of the attached mass, we first plot the frequency dependence of the wave travel coefficient in its absence at$M_1 = 0$. The corresponding plot is shown in Fig. 2. It can be seen that the maximum frequency range ($0 < \overline{k_1} \lesssim 1.75$), where the tunneling effect is manifested, is reached at$S_m = 0.34$. At higher values of the frequency$\overline{k}$, formula (12) transforms into a solution describing the transmission coefficient oscillations corresponding to the usual traveling wave mode.

If we take into account the attached mass and put$\gamma = \frac{M_2}{M_1} = 0.24$, the dependence of the wave transmission coefficient will take the form shown in Fig. 3. 3. As can be seen from Fig. 3, the effect of full tunneling in the frequency range in which it was observed in the absence of the attached mass is destroyed. The tunneling coefficient reaches unity only at selected frequencies. Nevertheless, the region where the transmission coefficient is sufficiently close to unity remains. In addition, for small values of the dimensionless minimum cross-sectional area, the transmittance increases compared to the case with no attached mass.

We plot the dependence of the square of the transmission coefficient on the dimensionless wave number as a function of the attached mass$M_1 = 0.14; 0.43; 0.88$ (curves 1, 2, 3 in Fig. 4, respectively) with the same dimensionless minimum cross-sectional area$S_m = 0.34$, in order to analyze how the field changes as the attached mass increases.

Thus, from Fig. 4, we can conclude that with the increase of the parameterM_1 the transmission coefficient decreases, i.e., the increase of the attached mass leads to the fact that the field is distorted more strongly.

The results obtained by show that, indeed, when taking into account the of the attached mass, the conditions of full transmission deteriorate, and frequency-dependent effects appear in the low-frequency region, where the tunneling plateau is located. Nevertheless, it is hoped that less signal distortion can be achieved in the propagation of low-frequency signals. This task requires further research.

ACCOUNTING FOR DISSIPATION AND NONLINEARITY

Speaking of frequency-dependent effects, one cannot fail to mention such an important factor as the viscosity of the medium. Undoubtedly, it will introduce distortions, including in the propagation of low-frequency signals. Keeping in mind the problem of high intensity wave propagation, we will note the following peculiarities within the framework of this work.

In problems related to the propagation of acoustic waves of high intensity, a generalized Webster-type equation arises. It differs from equation (1) by the presence of two additional terms that describe nonlinear and dissipative effects. Usually, this equation is written in the form:

$$\frac{1}{S(x)}\frac{\partial}{\partial x}\left(S(x)\frac{\partial p}{\partial x}\right)-\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}+\frac{\varepsilon}{c^4\rho}\frac{\partial^2 p^2}{\partial t^2}+\frac{b}{\rho c^2}\frac{\partial^3 p}{\partial t\partial x^2}=0. \qquad (13)$$

Hereε, b is a nonlinear parameter and dissipation coefficient. Webster's equation of the type (13), especially when solving nonlinear problems, can be simplified by using the slowly changing profile method [11,12]. This method is applicable when changes in cross-sectional area occur slowly. As a result, the order of the nonlinear equation (13) is reduced. Following the standard procedure, we pass from the variablesx, t in equation (13) to new independent variables: the "slow" coordinate$x_1 = \delta x$ (whereδ is a small parameter of the problem) and the "fast" time $\tau = t - \frac{x}{c}$ in a coordinate system running at the speed of sound. Restricting ourselves to the standard model (13) and neglecting small terms of the order ,δ^n $n \geq 2$, we arrive at the evolution equation

$$\frac{\partial p}{\partial x}-\frac{\varepsilon}{c^3\rho}p\frac{\partial p}{\partial \tau}-\frac{b}{2c^3\rho}\frac{\partial^2 p}{\partial \tau^2}+\frac{p}{2}\frac{\partial}{\partial x}\left(\ln S\left(x\right)\right)=0. \qquad (14)$$

It is usually noted that equations of the form (13) - (14) have a wider area of applicability and describe also, for example, the fields of intense confined acoustic beams in inhomogeneous media in the approximation of nonlinear geometric acoustics (NGA), and the function $S(x)$ in this case has the meaning of the cross-sectional area of the beam tube [12].

However, while for beams in the NGA approximation the notation of the dissipative term in equations (13) - (14) is exact, for waves in a tube it is, generally speaking, not so. A rigorous derivation of the generalized linearized Webster equation for a tube of variable cross section with viscosity taken into account leads to an exact closed equation for the oscillatory velocity:

$$\frac{\partial^2 u}{\partial t^2} = c_0^2 \frac{\partial}{\partial x}\left(\frac{1}{S}\frac{\partial}{\partial x}(Su)\right) + \frac{b}{\rho_0}\frac{\partial^3 u}{\partial t \partial x^2}, \qquad (15)$$

in which the viscous term has the usual structure as in equation (13), but the variable cross-sectional area term has a different structure compared to equation (1) for pressure. The transition to the closed equation for pressure is possible only approximately, taking into account those or other terms of different order of smallness. In the simplest variant we obtain the following approximate closed equation for pressure:

$$\frac{\partial^2 p}{\partial t^2} = \frac{1}{S}\left(c_0^2 + \frac{b}{\rho_0}\frac{\partial}{\partial t}\right)\frac{\partial}{\partial x}\left(S\frac{\partial p}{\partial x}\right). \qquad (16)$$

Here, the variable cross-sectional area term coincides with the analogous term in equation (1), but the viscous term has a different structure than in equation (13). Thus, the generalized Webster's equation with viscosity for sound in tubes is, generally speaking, different from the form (13), and the dissipative effects in tubes and in confined beams are also different. In addition, the difference in equations (13)-(16) may affect the shape of wave profiles formed in a tube of variable cross section and the possibility of finding accurate analytical solutions.

CONCLUSION

Thus, we have studied wave propagation in the tunneling mode in narrow tubes with a variable cross section of a special kind. The influence of the attached mass at the boundaries of the region of smooth narrowing of the tube has been taken into account. It was found out that taking into account the attached mass worsens the conditions of wave tunneling: this is manifested in the decrease of the wave passage coefficient in the frequency range, where the tunneling mode was observed without taking into account the attached mass, as well as in the reduction of the frequency range itself. However, when the attached mass is taken into account, the interval of minimum values of the cross-sectional area increases, at which almost complete wave passage is possible (the passage coefficient is close to unity). It is shown that an increase in the attached mass leads to a decrease in the wave transmission coefficient, thereby distorting the field even more.

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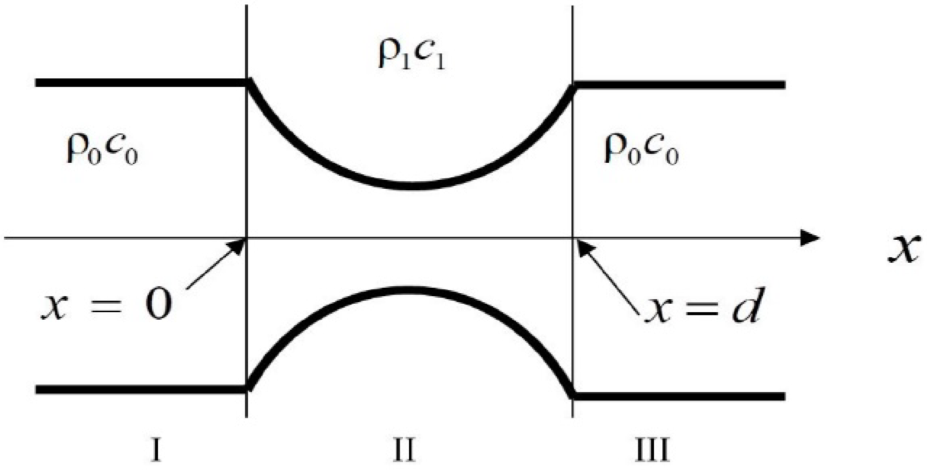
FIGURE CAPTIONS

Fig. 1. The tube contains a constriction region II (0 x< < d) filled with a medium with densityρ_1 and speedc_1 of sound.

Fig. 2. Frequency dependences of the square of the wave travel coefficient without taking into account the attached mass.

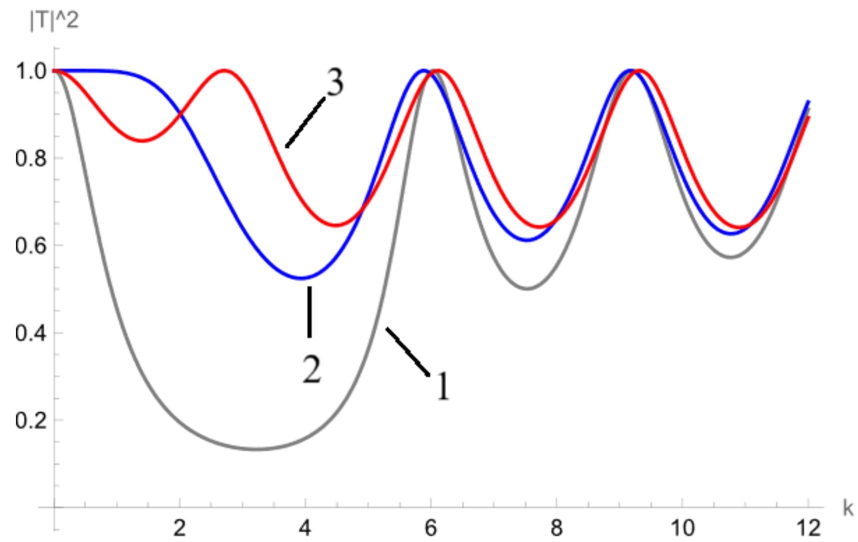
Fig. 3. Frequency dependences of the square of the wave travel coefficient taking into account the attached mass.

Fig. 4. Frequency dependences of the square of the wave-passing coefficient taking into account the attached mass at different .M_1



$\rho_1 c_1$
$\rho_0 c_0$
$\rho_0 c_0$
x
$x = 0$
$x = d$
I
II
III

Fig. 1.



|T|^2
1.0
0.8
0.6
0.4
0.2
2
4
6
8
10
12
k
1
2
3

Fig. 2.

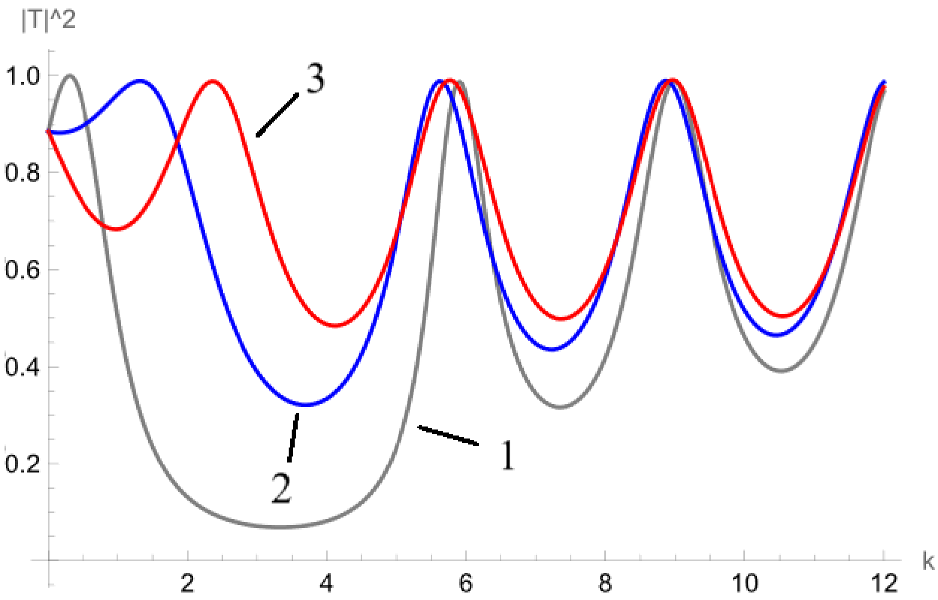


Fig. 3.

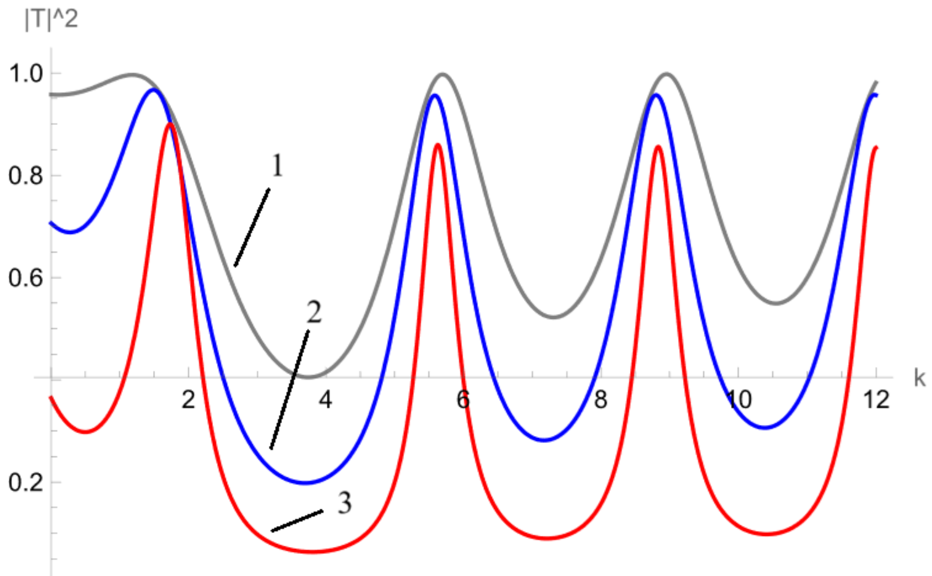


Fig. 4.