Pis'ma v ZhETF, vol. 117, iss. 3, pp. 248 – 249

Dimensionless physics: Planck constant as an element of Minkowski metric

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Submitted 27 November 2022
Resubmitted 19 December 2022
Accepted 19 December 2022

DOI: 10.31857/S1234567823030114, EDN: oxtlrm

Several approaches to quantum gravity (including Diakonov tetrads emerging as the bilinear combinations of the fermionic fields [1]; BF-theories of gravity; the model of superplastic vacuum; and effective acoustic metric) suggest [2, 3] that in general relativity the metric must have dimension 2, i.e. $[g_{\mu\nu}] = 1/[L]^2$. In particular, the model of the superplastic vacuum [4] is described in terms of the so-called elasticity tetrads [5–10] $E^a_\mu = \partial X^a/\partial x^\mu$, where equations $X^a(x) = 2\pi n_a$ are equations of the (deformed) crystal planes. Since the functions X^a play the role of the geometric $U(1)$ phases and thus are dimensionless, the elasticity tetrads play the role of the gauge fields (translation gauge fields). That is why tetrads have the same dimension 1 as the dimension of gauge fields: $[E^a_\mu] = 1/[L]$.

Originally the dimension 1 tetrads appeared in the Diakonov theory [1–13], where the tetrads emerge as bilinear combinations of the quantum fermionic fields:

$$E^a_\mu = \frac{1}{2}\left(\Psi^\dagger\gamma^a\partial_\mu\Psi - \Psi^\dagger\overleftarrow{\partial_\mu}\gamma^a\Psi\right) . \qquad (1)$$

In this approach to quantum gravity the metric and the space-time distance are both the quantum objects made of the leptons and quarks [14]. Here we suggest that the Diakonov theory leads to the unusual dimension of the Planck constant $\hbar$.

Diakonov tetrads and elasticity tetrads give rise to the covariant metric $g_{\mu\nu} = \eta_{ab}E^a_\mu E^b_\nu$ with dimension 2, $[g_{\mu\nu}] = 1/[L]^2$, and the contravariant metric with dimension -2, $[g^{\mu\nu}] = [L]^2$. The determinant of the tetrads has dimension 4, $[e] = [\sqrt{-g}] = 1/[L]^4$, while the interval is dimensionless, $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, $[s^2] = [1]$. Since the interval describes the classical dynamics of a point particle with action $S = M\int ds$, the particle mass M is dimensionless, $[M] = [1]$, as well as all other diffeomorphism invariant quantities, such as action S, interval s, cosmological constant Λ, scalar curvature R, scalar field Φ, etc. [2, 3]. The variation of action leads to the Hamilton–Jacobi equation $g^{\mu\nu}\partial_\mu S\partial_\nu S + M^2 = 0$, where both terms are dimensionless. Since mass is dimensionless, GM^2/r is dimensionless, which leads to the length dimension of Newton constant, $[G] = [L]$. The wave vector $k_\mu = \partial_\mu S$ has dimension $[k_\mu] = 1/[L]$ and obeys equation $g^{\mu\nu}k_\mu k_\nu + M^2 = 0$. In the flat Minkowski spacetime the wave vector obeys equation $|g^{\rm Mink}_{00}|^{-1}(\omega^2 - \mathbf{k}^2) = M^2$. This suggests that it is natural to identify the Planck constant $\hbar$ with the element of Minkowski metric $\hbar = 1/\sqrt{-g^{\rm Mink}_{00}}$ [15]. Then the red shift equation, $M_m - M_n = \sqrt{-g^{00}}\,\omega_{mn}$, becomes the generalization of the conventional relation between the energy levels and frequency of radiating photon in Minkowski vacuum, $E_m - E_n = \hbar\omega_{mn}$.

Note that as distinct from the dimensionless quantities, which are diffeomorphism invariant, the parameter $\hbar$ is not diffeomorphism invariant. It is determined only in the Minkowski vacuum, and being the element of the Minkowski metric it is invariant only under Lorentz transformations. As a result the Planck constant $\hbar$ is not dimensionless, and has the dimension of length [L]. Then according to Weinberg criterion [16] $\hbar$ cannot be the fundamental constant (see also [17–20] on fundamental constants).

Another parameter of Minkowski spacetime is the speed of light c, which enters the metric in the following way: $g^{\mu\nu}_{\rm Mink} = {\rm diag}(\hbar^2, \hbar^2c^2, \hbar^2c^2, \hbar^2c^2)$. This parameter is invariant only under space rotation group $SO(3)$, and has dimension $[L]/[t]$. If the parameter c is taken into account, the Planck constant has dimension of time $[\hbar] = [M][t] = [t]$. Except for the special cases, we use units with $c = 1$.

The parameter $\hbar$ enters only the Minkowski metric, and does not enter any equation written in the covariant form, i.e. in terms of the full metric. That is why in general the commutation relations for position and mo-

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mentum operators in quantum mechanics do not contain $\hbar$, $[\hat{k}_i, \hat{x}^j] = i\delta_i^j$. As a result, the elementary volume of phase space $\int dk\, dx$ and action are dimensionless, while the parameter $\hbar$ of Minkowski vacuum has dimension of time $[t]$.

The quadratic terms in the action for the classical scalar field Φ in the N-dimensional spacetime is:

$$S = \int d^N x \sqrt{-g} \left(g^{\mu\nu} \nabla_\mu \Phi^* \nabla_\nu \Phi + M^2 |\Phi|^2 \right). \quad (2)$$

Due to zero dimensions of metric and mass the scalar field is dimensionless, $[\Phi] = [1]$, which differs from the dimension $n = (N-2)/2$ of scalar fields in the conventional approach. Expanding the Klein–Gordon equation for scalar Φ over $1/M$ one obtains the nonrelativistic Schrödinger action. In Minkowski spacetime, introducing the Schrödinger wave function ψ: $\Phi(\mathbf{r}, t) = \frac{1}{\sqrt{M}} \exp\left(iMt/\sqrt{-g^{00}} \right) \psi(\mathbf{r}, t)$, one obtains the Schrödinger-type action

$$S_{\rm Schr} = \int d^3x dt \sqrt{-g}\, \mathcal{L}, \quad (3)$$

$$2\mathcal{L} = i\sqrt{-g^{00}} \left(\psi \partial_t \psi^* - \psi^* \partial_t \psi \right) + \frac{g^{ik}}{M} \nabla_i \psi^* \nabla_k \psi. \quad (4)$$

In Minkowski vacuum $\sqrt{-g^{00}_{\rm Mink}} \equiv \hbar$, $g^{ik}_{\rm Mink} \equiv \hbar^2 \delta^{ik}$, one obtains the conventional Schrödinger wave equation:

$$i\hbar \partial_t \psi = -\frac{\hbar^2}{2M} \nabla^2 \psi. \quad (5)$$

This is another consequence of the metric with dimension $1/[L]^2$: the quantum mechanical Schrödinger equation for nonrelativistic particle is obtained from the classical relativistic scalar field.

While temperature is dimensionless, $[T] = [M] = [1]$, the Tolman temperature $T_{\rm Tolman} = T(r)\sqrt{-g_{00}(r)}$ has dimension of inverse length, $[T_{\rm Tolman}] = 1/[L]$. The parameter $\hbar$ determines the ratio between the temperature and Tolman temperature in the Minkowski vacuum. In principle there can be different Minkowski vacua, with cosmological phase transitions between these vacua [21]. Then each vacuum may have its own value of the parameter $\hbar$. In the thermal contact between the two vacua they must have the same Tolman temperature, and thus their temperatures obey the rule, $\hbar_1/T_1 = \hbar_2/T_2$. This means that in thermal equilibrium the contacting Minkowski vacua have the same time τ on imaginary axis, $\tau_1 = \tau_2$.

The Planck length scale has the conventional form $l_{\rm P}^2 = \hbar G$, with $[l_{\rm P}]^2 = [\hbar][G] = [L][L] = [L]^2$. The Planck constant has the same dimension as the Planck length, $[\hbar] = [l_{\rm P}] = [L]$. Whether this "Planck constant length" is related to the "Planck length scale", is an open question [22]. Anyway, the Diakonov theory [1] suggests the close connection between gravity and quantum mechanics. The dimension $1/[L]^2$ of the metric suggests that such metric describes the dynamics, quantum mechanics and thermodynamics, rather than the geometry.

I thank Y. N. Obukhov and N. N. Nikolaev for discussion and criticism.

This work has been supported by the European Research Council (ERC) under the European Union's Horizon 2020 programme (Grant Agreement # 694248).

This is an excerpt of the article "Dimensionless physics: Planck constant as an element of Minkowski metric". Full text of the paper is published in JETP Letters journal. DOI: 10.1134/S0021364022603013

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1. D. Diakonov, arXiv:1109.0091.
2. G. E. Volovik, JETP **132**, 727 (2021).
3. G. E. Volovik, ZhETF **162**, 680 (2022).
4. F. R. Klinkhamer and G. E. Volovik, JETP Lett. **109**, 364 (2019).
5. I. E. Dzyaloshinskii and G. E. Volovick, Ann. Phys. **125**, 67 (1980).
6. J. Nissinen and G. E. Volovik, Phys. Rev. Research **1**, 023007 (2019).
7. J. Nissinen, Ann. Phys. **447**, 169139 (2022).
8. J. Nissinen, Phys. Rev. Lett. **124**, 117002 (2020).
9. S. Laurila and J. Nissinen, Phys. Rev. B **102**, 235163 (2020).
10. L. Gioia, Ch. Wang, and A. A. Burkov, Phys. Rev. Research **3**, 043067 (2021).
11. A. A. Vladimirov and D. Diakonov, Phys. Rev. D **86**, 104019 (2012).
12. A. A. Vladimirov and D. Diakonov, Physics of Particles and Nuclei **45**, 800 (2014).
13. Y. N. Obukhov and F. W. Hehl, Phys. Lett. B **713**, 321 (2012).
14. K. Akama, Progress of Theoretical Physics **60**, 1900 (1978).
15. G. E. Volovik, JETP Lett. **90**, 697 (2009).
16. S. Weinberg and J. G. Taylor, Philos. Trans. R. Soc. London A **310**, 249 (1983).
17. M. J. Duff, L. B. Okun, and G. Veneziano, JHEP **0203**, 023 (2002).
18. L. B. Okun, *Cube or hypercube of natural units, in Multiple facets of quantization and supersymmetry, Michael Marinov Memorial Volume*, ed. by M. Olshanetsky and A. Vainshtein, World Scientific, Singapore (2002).
19. L. B. Okun, Physics of Atomic Nuclei **65**, 1370 (2002).
20. G. Gamow, D. Ivanenko, and L. Landau, Physics of Atomic Nuclei **65**, 1373 (2002).
21. F. R. Klinkhamer, Phys. Rev. D **106**, 124015 (2022).
22. S. Carlip, arXiv:2209.14282.