Известия Саратовского университета. Новая серия. Серия: Математика. Механика. Информатика. 2025. Т. 25, вып. 2. С. 167–172
Izvestiya of Saratov University. Mathematics. Mechanics. Informatics, 2025, vol. 25, iss. 2, pp. 167–172
https://mmi.sgu.ru https://doi.org/10.18500/1816-9791-2025-25-2-167-172, EDN: ENEECH

Article

Estimation of the difference of partial sums of expansions by the root functions of the differential operator and into trigonometric Fourier series

V. S. Rykhlov

Saratov State University, 83 Astrakhanskaya St., Saratov 410012, Russia

Victor S. Rykhlov, rykhlovvs@yandex.ru, https://orcid.org/0000-0003-1556-7707, SPIN: 5650-4265, AuthorID: 2936

Abstract. We consider a linear ordinary differential operator defined by an n-th order differential expression with a nonzero coefficient for $(n-1)$th derivative and Birkhoff regular two-point boundary conditions. The question of the uniform convergence of expansions of a function into a series of root functions of the operator L and the usual trigonometric Fourier series, as well as the estimation of the difference of the corresponding partial sums, is investigated. Estimates of the difference of the partial sums of these expansions are obtained in terms of the general (integral) modules of continuity of the expandable function and the coefficient at the $(n-1)$th derivative. The proof essentially uses the estimate (previously obtained by the author) of the difference between the partial sums of the expansions of a function in a series with respect to the root functions of the operator L and in the modified trigonometric Fourier series, as well as the author's analogue of the Steinhaus theorem in terms of general modules of continuity.

Keywords: ordinary differential operator, root functions, eigen and associated functions, expansion in a series by root functions, equiconvergence of expansions, estimation of the difference of partial sums, integral modules of continuity.

For citation: Rykhlov V. S. Estimation of the difference of partial sums of expansions by the root functions of the differential operator and into trigonometric Fourier series. *Izvestiya of Saratov University. Mathematics. Mechanics. Informatics*, 2025, vol. 25, iss. 2, pp. 167–172. https://doi.org/10.18500/1816-9791-2025-25-2-167-172, EDN: ENEECH

Научная статья
УДК 517.927.25

Оценка разности частичных сумм разложений по корневым функциям дифференциального оператора и в тригонометрический ряд Фурье

В. С. Рыхлов

Саратовский национальный исследовательский государственный университет имени Н. Г. Чернышевского, Россия, 410012, г. Саратов, ул. Астраханская, д. 83

Рыхлов Виктор Сергеевич, кандидат физико-математических наук, доцент кафедры дифференциальных уравнений и математической экономики, rykhlovvs@yandex.ru, https://orcid.org/0000-0003-1556-7707, SPIN: 5650-4265, AuthorID: 2936

Аннотация. Рассматривается линейный обыкновенный дифференциальный оператор, определяемый дифференциальным выражением n-го порядка с ненулевым коэффициентом при $(n-1)$-й производной и регулярными по Биркгофу двухточечными краевыми условиями. Исследуется вопрос о равномерной равносходимости разложений функции в ряд по корневым функциям оператора L и в обычный тригонометрический ряд Фурье, а также об оценке разности соответствующих частичных сумм. Получены оценки разности частичных сумм этих разложений в терминах общих (интегральных) модулей непрерывности разлагаемой функции и коэффициента при $(n-1)$-й производной. Доказательство существенно использует ранее полученную автором оценку разности частичных сумм разложений функции в ряд по корневым функциям оператора L и в модифицированный тригонометрический ряд Фурье, а также полученный автором аналог теоремы Штейнгауза в терминах общих модулей непрерывности.

Ключевые слова: обыкновенный дифференциальный оператор, корневые функции, собственные и присоединенные функции, разложение в ряд по корневым функциям, равносходимость разложений, оценка разности частичных сумм, интегральные модули непрерывности

Для цитирования: *Rykhlov V. S.* Estimation of the difference of partial sums of expansions by the root functions of the differential operator and into trigonometric Fourier series [*Рыхлов В. С.* Оценка разности частичных сумм разложений по корневым функциям дифференциального оператора и в тригонометрический ряд Фурье] // Известия Саратовского университета. Новая серия. Серия: Математика. Механика. Информатика. 2025. Т. 25, вып. 2. С. 167–172. https://doi.org/10.18500/1816-9791-2025-25-2-167-172, EDN: ENEECH

Introduction

Consider the differential operator L generated by the differential expression

$$\ell(y) = y^{(n)} + p_1(x)y^{(n-1)} + ... + p_n(x)y, \qquad p_j(x) \in L_1[0,1],$$

and boundary conditions

$$\sum_{j=0}^{n-1}(a_{kj}y^{(j)}(0) + b_{kj}y^{(j)}(1)) = 0, \qquad k = \overline{1,n}. \tag{1}$$

Suppose that the boundary conditions (1) are Birkhoff regular [1, pp. 66–67].

One of the important problems is the problem of expansion of a given function into a series of root functions (r.f.) of the operator L. This problem is most completely solved in the case when it is possible to prove the equiconvergence of the expansion (in one or another sense) of the given function into the series on the r.f. of the operator L and into the trigonometric Fourier series since the trigonometric system is well enough studied.

The trigonometric system is a system of the r.f. of the operator L_0 of the form

$$\ell_0(y) = y^{(n)}, \quad y^{(k-1)}(0) - y^{(k-1)}(1) = 0, \quad k = \overline{1,n}.$$

We study the influence of the properties of the coefficient $p_1(x)$ and the expandable function $f(x)$ on the estimation of the difference of partial sums of series expansions by the r.f. of operators L and L_0 in the uniform metric inside the interval $(0,1)$.

Some results have already been published in [2–4]. A brief background can also be found there. A more detailed background is given in the review articles [5, 6].

1. Basic concepts. Formulation of the theorem

Let λ_ν, $\lambda_{0\nu}$, $\nu = 0,1,\ldots$, be the eigenvalues (e.v.) of the operators L and L_0, respectively. Let us denote by D_δ the region of $\mathbb{C}$ resulting from the removal of all e.v. operators L and L_0 together with circles with centers in the e.v. and of sufficiently small fixed radius $\delta > 0$.

From the asymptotics of e.v. [1, p. 74–75] it follows that there exists a sequence $\{r_m\}_{m=1}^{\infty} \in \mathbb{R}$ such that $(2\pi m)^n < r_m < (2\pi(m+1))^n$, and the contours $\Gamma_m := \{\lambda \in \mathbb{C} : |\lambda| = r_m\}$, starting from some sufficiently large m, lie in the region D_δ.

Between neighboring contours, there are at most two e.v. λ_ν of the operator L starting from some m, and one double e.v. $\lambda_{0\mu}$ of the operator L_0. The consideration of such contours is due to the structure of the usual trigonometric Fourier series, which is actually a series with brackets.

Let us denote by R_λ and $R_{0\lambda}$ the resolvents of the operators L and L_0, respectively. Let

$$S_m(f,x) = -\frac{1}{2\pi i}\int\limits_{\Gamma_m} R_\lambda f\, d\lambda, \quad \sigma_m(f,x) \equiv S_{0m}(f,x) = -\frac{1}{2\pi i}\int\limits_{\Gamma_m} R_{0\lambda} f\, d\lambda.$$

It is known [1, p. 92] that $S_m(f,x)$ is a partial sum of the biorthogonal Fourier series of the function $f(x)$ by the r.f. of the operator L, containing summands corresponding to the e.v. λ_ν for which $|\lambda_\nu| < r_m$, and $\sigma_m(f)$ is the partial sum of order m of the ordinary trigonometric Fourier series of the function $f(x)$ (the partial sum of the orthogonal Fourier series of the function $f(x)$ by the r.f. of the operator L_0, containing summands corresponding to the e.v. $\lambda_{0\nu} = (2\nu\pi i)^n$ for which $|\nu| \leqslant m$), i.e.

$$\sigma_m(f,x) := \sum_{k=-m}^{m} (f,e_k)e_k(x), \text{ where } (f,e_k) := \int_0^1 f(\xi)e^{-2k\pi i\xi}\, d\xi.$$

For $f(x)$ from $L_1[0,1]$ denote

$$\Phi_m(x) = S_m(f,x) - V\sigma_m(V^{-1}f,x), \quad \Psi_m(x) = S_m(f,x) - \sigma_m(f,x), \tag{2}$$

where

$$V(x) = \exp\biggl(-\frac{1}{n}\int\limits_0^x p_1(\tau)\, d\tau\biggr). \tag{3}$$

Let us introduce the following continuity modules

$$\omega(f,\delta)_p = \sup_{0<h\leqslant\delta}\left(\int\limits_0^{1-h} |f(t+h) - f(t)|^p\, dt\right)^{1/p}, \text{ где } 1 \leqslant p < \infty,$$

$$\omega(f,\delta)_\infty = \sup_{0<h\leqslant\delta}\ \sup_{t\in[0,1-h]} \bigl|f(t+h) - f(t)\bigr|.$$

By $L_\infty[0,1]$, we understand the space $C[0,1]$.

The following condition is used further

$$f(x) \in L_r[0,1], \quad p_1(x) \in L_q[0,1], \quad \frac{1}{q} + \frac{1}{r} = 1, \quad 1 \leqslant q \leqslant \infty. \tag{4}$$

To formulate the theorem on the estimation of $\Psi_m(x)$ we introduce the following notations:

— for the function $h(x) \in C[0,1]$ denote by

$$H_q(h,m) := \left(\int\limits_{1/m}^{1} \frac{1}{\xi}\, h^q(\xi)\, d\xi\right)^{1/q} \quad \text{when } 1 \leqslant q < \infty, \quad H_\infty(h,m) := 1;$$

— for the function $g(x) \in L_q[0,1]$ denote by

$$\theta_q(g,\delta) := \sup_{\tau\in[0,\delta]} \theta_{1q}(g,\tau) \text{ when } 1 \leqslant q < \infty \text{ and } \theta_\infty(g,\delta) \equiv 1,$$

$$\theta_{1q}(g,\tau) := \inf_{k\in\mathbb{N}} \left\{ \omega\Big(g, \frac{1}{k}\Big)_q + (k^2\tau)^{\frac{1}{q}} \right\} \quad \text{when} \quad 1 \leqslant q < \infty,$$

$$\hat{\theta}_q(g,\delta) = \sup_{\tau\in[0,\delta]} \hat{\theta}_{1q}(g,\tau) \text{ when } 1 \leqslant q < \infty \text{ and } \hat{\theta}_\infty(g,\delta) \equiv 1,$$

$$\hat{\theta}_{1q}(g,\tau) = \inf_{k\in\mathbb{N}} \left\{ \omega\Big(g, \frac{1}{k}\Big)_q + \frac{1}{k^{1/r}}\theta_q\left(g, \frac{1}{k}\right) + (k^2\tau)^{\frac{1}{q}} \right\} \quad \text{when} \quad 1 \leqslant q < \infty.$$

Theorem 1. *Under the condition* (4) *for any compact* $K \subset (0,1)$ *and* $m \in \mathbb{N}$, $m \gg 1$,

$$\|\Psi_m(x)\|_{C(K)} \leqslant C(f,L,K)B_m$$

where

$$B_m = \inf_{k\in\mathbb{N}} \left\{ \left(\ln m\,\omega\left(p_1, \frac{1}{k}\right)_q + 1 + H_q\big(\hat{\theta}_q(p_1,\cdot), m\big) + \frac{k^{2/q}}{m^{1/q}} + \frac{k^2\ln m}{m} \right) \omega\left(f, \frac{1}{k}\right)_r + \right.$$
$$\left. + \left(1 + \frac{k^2\ln m}{m}\right)\omega\left(p_1, \frac{1}{k}\right)_q + \frac{1}{k^{1/r}}\theta_q\left(p_1, \frac{1}{k}\right) + \frac{k^{2+2/q}}{m^{1+1/q}} + \frac{k^4\ln m}{m} \right\}.$$

Theorem 1 was previously formulated in [4], but the proof was not given. This paper fills this gap. Moreover, the formulation of Theorem 1 clarifies the formulation of the corresponding theorem from [4].

If we impose conditions on $\omega(f,\delta)_r$ and $\omega(p_1,\delta)_q$ such that $B_m \to 0$ at $m \to \infty$, we can obtain various sufficient conditions of equiconvergence of the corresponding expansions with partial sum difference estimates (see [2–4]).

2. Proof of the theorem 1

The validity of the statement of Theorem 1 in the case of general continuity modules follows from Theorem 1 [4] and the analog of Steinghaus' theorem [7] for general continuity modules on the basis of the relation

$$\Psi_m(x) = S_m(f,x) - \sigma_m(f,x) = S_m(f,x) - V(x)\sigma_m(f,x) +$$
$$+V(x)\Big(\sigma_m(V^{-1}f,x) - V^{-1}(x)\sigma_m(f,x)\Big) = \Phi_m(x) + V(x)\Big(\sigma_m(V^{-1}f,x) - V^{-1}(x)\sigma_m(f,x)\Big) =$$
$$= \Phi_m(x) - V(x)\Big(\mathcal{W}(x)\sigma_m(f,x) - \sigma_m(\mathcal{W}f,x)\Big) = \Phi_m(x) - V(x)\Theta_m(x), \quad (5)$$

where the notations (2) are used, $V(x)$ is defined by the formula (3), $\mathcal{W}(x) := V^{-1}(x)$ and $\Theta_m(x) := \mathcal{W}(x)\sigma_m(f,x) - \sigma_m(\mathcal{W}f,x)$.

The following formula is valid for the function $\mathcal{W}(x)$:

$$\mathcal{W}(x) = V^{-1}(x) = \exp\left(\frac{1}{n}\int_0^x p_1(\tau)\,d\tau\right) = 1 + \int_0^x \tilde{g}(t)\,dt, \quad (6)$$

where

$$\tilde{g}(t) := \frac{1}{n}p_1(t)V^{-1}(t). \quad (7)$$

Clearly, if $p_1(t) \in L_q[0,1]$, then so does $\tilde{g}(t) \in L_q[0,1]$ and vice versa. Hence, the function $\mathcal{W}(x)$ is the analog of the function $W(x)$ in the analog of Steingaz's theorem from [7] for the case of general continuity modules and for it instead of the formula

$$W(x) = \mathcal{W}_0 + \int\int_0^x g(t)\,dt,$$

the formula (6) with $\tilde{g}(t) \in L_q[0,1]$ is valid.

Then, based on Theorem 1, we obtain from [7] the following estimate

$$\|\Theta_m(x)\|_{C(K)} \leqslant C(f, \tilde{g}, K) Q_m \tag{8}$$

where

$$Q_m = \inf_{k \in \mathbb{N}} \left\{ \omega\left(f, \frac{1}{k}\right)_r H_q\big(\theta_q(\tilde{g}, \cdot), m\big) + \omega\left(\tilde{g}, \frac{1}{k}\right)_q + \frac{k^4 \ln m}{m} \right\}. \tag{9}$$

Let us evaluate all objects containing the function $\tilde{g}(x)$ through similar objects containing only the original function $p_1(x)$.

If we denote $v := \max\limits_{t \in [0,1]} |V^{-1}(t)|$, then based on (6) we obtain

$$\begin{aligned}
\omega(\tilde{g}, \delta)_q &= \frac{1}{n} \sup_{0<h\leqslant\delta} \left(\int_0^{1-h} |p_1(t+h)V^{-1}(t+h) - p_1(t)V^{-1}(t)|^q \, dt \right)^{1/q} \leqslant \\
&\leqslant \frac{1}{n} \left(\sup_{0<h\leqslant\delta} \left(\int_0^{1-h} |p_1(t+h) - p_1(t)|^q |V^{-1}(t+h)|^q \, dt \right)^{1/q} + \right. \\
&\left. + \sup_{0<h\leqslant\delta} \left(\int_0^{1-h} |p_1(t)|^q |V^{-1}(t+h) - V^{-1}(t)|^q \, dt \right)^{1/q} \right) \leqslant \\
&\leqslant \frac{v}{n} \omega(p_1, \delta)_q + \|p_1(x)\|_q \sup_{0<h\leqslant\delta} \sup_{t\in[0,1-h]} |V^{-1}(t+h) - V^{-1}(t)| = \\
&= \frac{v}{n} \omega(p_1, \delta)_q + \|p_1(x)\|_q \sup_{0<h\leqslant\delta} \sup_{t\in[0,1-h]} \left| \int_0^{1-h} \tilde{g}(\tau) \, d\tau \right|.
\end{aligned} \tag{10}$$

Since $p_1(x) \in L_q[0,1]$, based on the equality (7) we have from [7]

$$\int_t^{t+h} |\tilde{g}(\tau)| \, d\tau \leqslant \frac{v}{n} \int_t^{t+h} |p_1(\tau)| \, d\tau \leqslant C(p_1) h^{\frac{1}{r}} \theta_q(p_1, h). \tag{11}$$

It follows from (10), (11) that

$$\omega(\tilde{g}, \delta)_q \leqslant C(p_1) \left(\omega(p_1, \delta)_q + \delta^{\frac{1}{r}} \theta_q(p_1, \delta) \right). \tag{12}$$

Using this estimate, we obtain

$$\theta_{1q}(\tilde{g}, \tau) \leqslant C(p_1) \inf_{k \in \mathbb{N}} \left\{ \omega\left(p_1, \frac{1}{k}\right)_q + \frac{1}{k^{1/r}} \theta_q\left(p_1, \frac{1}{k}\right) + (k^2 \tau)^{\frac{1}{q}} \right\} = C(p_1) \hat{\theta}_{1q}(p_1, \tau).$$

Based on this estimate, we have

$$\theta_q(\tilde{g}, \xi) \leqslant C(p_1) \sup_{\tau \in [0,\xi]} \hat{\theta}_{1q}(p_1, \tau) = C(p_1) \hat{\theta}_q(p_1, \xi). \tag{13}$$

Finally, from (13) we obtain

$$H_q\big(\theta_q(\tilde{g}, \cdot), m\big) \leqslant C(p_1) \left(\int_{1/m}^1 \frac{1}{\xi} \hat{\theta}_q^q(p_1, \xi) \, d\xi \right)^{1/q} = C(p_1) H_q\big(\hat{\theta}_q(p_1, \cdot), m\big). \tag{14}$$

Thus, from (8), (9), (12), (14), it follows that the following estimation is valid

$$Q_m \leqslant C(f, p_1, K)\hat{Q}_m \tag{15}$$

where

$$\hat{Q}_m = \inf_{k\in\mathbb{N}} \left\{ H_q\big(\hat{\theta}_q(p_1,\cdot), m\big)\,\omega\left(f, \frac{1}{k}\right)_r + \omega\left(p_1, \frac{1}{k}\right)_q + \frac{1}{k^{1/r}}\theta_q\left(p_1, \frac{1}{k}\right) + \frac{k^4 \ln m}{m} \right\}. \tag{16}$$

Hence, based on the relations (5), (8), (15), (16), and Theorem 1 of [4] on the evaluation of $\|\Phi_m(x)\|$, we obtain the statement of Theorem 1.

Thus, Theorem 1 is proved.

References

1. Naimark M. A. *Lineynye differentsial'nye operatory* [Linear differential operators]. Moscow, Nauka, 1969. 527 p. (in Russian).
2. Rykhlov V. S. The rate of equiconvergence for differential operators with nonzero coefficient multiplying the $(n-1)$th derivative. *Doklady Akademii Nauk SSSR*, 1984, vol. 279, iss. 5, pp. 1053—1056 (in Russian).
3. Rykhlov V. S. Rate of equiconvergence for differential operators with nonzero coefficient of the $n-1$-th derivative. *Differential Equations*, 1990, vol. 26, iss. 6, pp. 704–715.
4. Rykhlov V. S. Equiconvergence rate in terms of general moduli of continuity for differential operators. *Results in Mathematics*, 1996, vol. 29, pp. 153–168. https://doi.org/10.1007/BF03322215
5. Minkin A. M. Equiconvergence theorems for differential operators. *Journal of Mathematical Sciences*, 1999, vol. 96, no. 6, pp. 3631–3715. https://doi.org/10.1007/bf02172664
6. Lomov I. S. Estimates of speed of convergence and equiconvergence of spectral decomposition of ordinary differential operators. *Izvestiya of Saratov University. Mathematics. Mechanics. Informatics*, 2015, vol. 15, iss. 4, pp. 405–418 (in Russian). https://doi.org/10.18500/1816-9791-2015-15-4-405-418
7. Rykhlov V. S. On the rate of equiconvergence in an analogue of the Steinghaus theorem. *Taurida Journal of Computer Science Theory and Mathematics*, 2015, iss. 3 (28), pp. 62–81 (in Russian).

Поступила в редакцию / Received 11.12.2024
Принята к публикации / Accepted 09.01.2025
Опубликована онлайн / Published online 30.05.2025