MATHEMATICS
https://doi.org/10.26117/2079-6641-2024-48-3-7-19
Research Article
Full text in English
MSC 35A08, 35J05, 35J15, 35J25

# Boundary Value Problems for the Three-Dimensional Helmholtz Equation in the Unbounded Octant, Square and Half Space

***Z. O. Arzikulov****

Fergana Polytechnic Institute, 150107, Fergana, Ferganskaya str., 86, Uzbekistan

**Abstract.** At present, the results of the study of boundary value problems for the two-dimensional Helmholtz equation with one and two singular coefficients are known. In the presence of two positive singular coefficients in the two-dimensional Helmholtz equation, explicit solutions of the Dirichlet, Neumann and Dirichlet-Neumann problems in a quarter plane are expressed through a confluent hypergeometric function of two variables. The established properties of the confluent hypergeometric function of two variables allow us to prove the theorem of uniqueness and existence of a solution to the problems posed.In this paper, we study the Dirichlet, Neumann, and Dirichlet-Neumann problems for the three-dimensional Helmholtz equation at zero values of singular coefficients in an octant, a quarter of space, and a half-space. Uniqueness and existence theorems are proved under certain restrictions on the data. The uniqueness of solutions of which is proved using the extremum principle for elliptic equations. Using the known fundamental (singular) solution of the Helmholtz equation, solutions to the problems under study are written out in explicit forms.

*Key words: confluent hypergeometric function of three variables; system of partial differential equations; asymptotic formula; three-dimensional Helmholtz equation with three singular coefficients; Dirichlet problem in the first infinite octant.*

Received: 27.09.2024; Revised: 26.10.2024; Accepted: 05.11.2024; First online: 20.11.2024

---

**For citation.** Arzikulov Z. O. Boundary value problems for the three-dimensional Helmholtz equation in the unbounded octant, square and half space. *Vestnik KRAUNC. Fiz.-mat. nauki.* 2024, **48**: 3, 7-19. EDN: MRZFAU. https://doi.org/10.26117/2079-6641-2024-48-3-7-19

**Funding.** The work was carried out without the support of funds.

**Competing interests.** There are no conflicts of interest regarding authorship and publication.

**Contribution and Responsibility.** Author is solely responsible for providing the final version of the article in print.

***Correspondence:** E-mail: zafarbekarzikulov1984@gmail.com

МАТЕМАТИКА
https://doi.org/10.26117/2079-6641-2024-48-3-7-19
Научная статья
Полный текст на английском языке
УДК 517.958

# Краевые задачи для трехмерного уравнения Гельмгольца в неограниченном октанте, квадрате и полупространстве

*З. О. Арзикулов**

Ферганский политехнический институт, 150107, ул. Ферганская, 86, г. Фергана, Узбекистан

**Аннотация.** В настоящее время известны результаты исследования краевых задач для двумерного уравнения Гельмгольца с одним и двумя сингулярными коэффициентами. При наличии двух положительных сингулярных коэффициентов в двумерном уравнении Гельмгольца явные решения задач Дирихле, Неймана и Дирихле-Неймана в четверти плоскости выражаются через вырожденную гипергеометрическую функцию двух переменных. Установленные свойства вырожденной гипергеометрической функции двух переменных позволяют доказать теорему единственности и существования решения поставленных задач. В данной работе изучаются задачи Дирихле, Неймана и Дирихле-Неймана для трехмерного уравнения Гельмгольца при нулевых значениях сингулярных коэффициентов в октанте, четверти пространства и полупространстве. Доказываются теоремы единственности и существования при определенных ограничениях на данные. Единственность решений которых доказывается с помощью принципа экстремума для эллиптических уравнений. Используя известное фундаментальное (сингулярное) решение уравнения Гельмгольца, решения исследуемых задач выписываются в явном виде.

*Ключевые слова: вырожденная гипергеометрическая функция трех переменных; система уравнений в частных производных; асимптотическая формула; трехмерное уравнение Гельмгольца с тремя сингулярными коэффициентами; задача Дирихле в первом бесконечном октанте*

Получение: 27.09.2024; Исправление: 26.10.2024; Принятие: 05.11.2024; Публикация онлайн: 20.11.2024

**Для цитирования.** Arzikulov Z. O. Boundary value problems for the three-dimensional Helmholtz equation in the unbounded octant, square and half space // *Вестник КРАУНЦ. Физ.-мат. науки*. 2024. Т. 48. № 3. C. 7-19. EDN: MRZFAU. https://doi.org/10.26117/2079-6641-2024-48-3-7-19.

**Финансирование.** Работа была выполнена без поддержки фондов

**Конкурирующие интересы.** Конфликтов интересов в отношении авторства и публикации нет.

**Авторский вклад и ответственность.** Автор несет ответственность за предоставление окончательной версии статьи в печать.

***Корреспонденция:** E-mail: zafarbekarzikulov1984@gmail.com

## Introduction

It's known, that the Helmholtz equation has a variety of applications in physics and other sciences, including the wave equation, the diffusion equation, and the Schrödinger equation for a free particle.

The Helmholtz equation often arises in the study of physical problems involving partial differential equations (PDEs) in both space and time. The Helmholtz equation, which represents a time-independent form of the wave equation, results from applying the technique of separation of variables to reduce the complexity of the analysis [15].

The two-dimensional analogue of the vibrating string is the vibrating membrane, with the edges clamped to be motionless. The Helmholtz equation was solved for many basic shapes in the 19th century: the rectangular membrane by Siméon Denis Poisson in 1829, the equilateral triangle by Gabriel Lamé in 1852, and the circular membrane by Alfred Clebsch in 1862. The elliptical drumhead was studied by Émile Mathieu, leading to Mathieu's differential equation.

Two- and more-dimensional Helmholtz equations

$$\sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} + \mu u = 0$$

and their related boundary-value problems have been investigated in a large number of papers [1–3, 12–14].

On the other hand, the equation has important applications. In 1952 Kapilevich [18] has solved Dirichlet and Neumann problems for multidimensional Helmholtz equation with singular coefficient

$$\sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} + \frac{2\alpha}{x_1}\frac{\partial u}{\partial x_1} + \mu u = 0, \;\; 0 < 2\alpha < 1$$

in the half-space. In 1978 Marichev [19] has investigated two-dimensional Helmholtz equation with two singular coefficients

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\alpha}{x}\frac{\partial u}{\partial x} + \frac{2\beta}{y}\frac{\partial u}{\partial y} + \mu u = 0, \;\; 0 < 2\alpha, \;\; 2\beta < 1. \tag{1}$$

There are many works [6–9, 11] devoted to the Helmholtz equation (1). For instance, in the work [10] the Dirichlet problem fot the singular Helmholtz equation (1) for $\mu = -\lambda^2$ is solved explicitly.

Generally speaking, our further goal is to pose and investigate boundary value problems for Helmholtz equation with three singular coefficients

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{2\alpha}{x}\frac{\partial u}{\partial x} + \frac{2\beta}{y}\frac{\partial u}{\partial y} + \frac{2\gamma}{y}\frac{\partial u}{\partial z} + \mu u = 0, \;\; 0 \le 2\alpha, \;\; 2\beta, 2\gamma < 1 \tag{2}$$

in some infinite domains.

For beginning, in the present paper, we study the Dirichlet, Neumann and Dirichlet-Neumann boundary value problems for equation (2) at $\alpha = \beta = \gamma = 0$ and $\mu = -\lambda^2$ in the unbounded domains – in an octant, square of the space and half-space.

## The Dirichlet problem $D_3^3$ for the Helmholtz equation in the first octant

Let us consider the following Helmholtz equation

$$u_{xx} + u_{yy} + u_{zz} - \lambda^2 u = 0, \qquad (3)$$

in the infinite domain $\Omega_3 \equiv \{(x, y, z) : x > 0,\ y > 0,\ z > 0\}$

**The Dirichlet problem** $D_3^3$. Find a regular solution $u(x, y, z)$ to the Helmholtz equation (3) in the class of functions $C\left(\overline{\Omega_3}\right) \cup C^2(\Omega_3)$, satisfying the conditions

$$u(x, y, 0) = \tau_1(x, y), \quad 0 \le x, y < \infty, \qquad (4)$$

$$u(x, 0, z) = \tau_2(x, z), \quad 0 \le x, z < \infty, \qquad (5)$$

$$u(0, y, z) = \tau_3(y, z), \quad 0 \le y, z < \infty, \qquad (6)$$

$$\lim_{R\to\infty} u(x, y, z) = 0, \ \ R = \sqrt{x^2 + y^2 + z^2}, \qquad (7)$$

where $\tau_k(t, s)$ are given functions such that

$$\tau_1(x, y) = \frac{\tilde{\tau}_1(x, y)}{(1 + x^2 + y^2)^{(1+\varepsilon_1)/2}},\ \tilde{\tau}_1(x, y) \in C(0 \le x, y < \infty),\ \varepsilon_1 > 0, \qquad (8)$$

$$\tau_2(x, z) = \frac{\tilde{\tau}_2(x, z)}{(1 + x^2 + z^2)^{(1+\varepsilon_2)/2}},\ \tilde{\tau}_2(x, z) \in C(0 \le x, z < \infty),\ \varepsilon_2 > 0, \qquad (9)$$

$$\tau_3(y, z) = \frac{\tilde{\tau}_3(y, z)}{(1 + y^2 + z^2)^{(1+\varepsilon_3)/2}},\ \tilde{\tau}_3(y, z) \in C(0 \le y, z < \infty),\ \varepsilon_3 > 0. \qquad (10)$$

In addition, the functions $\tau_1(x, y)$, $\tau_2(x, z)$ and $\tau_3(y, z)$ satisfy the concordance conditions in origin $\tau_1(0, 0) = \tau_2(0, 0) = \tau_3(0, 0)$, and in the lateral ribs of the domain $\Omega_3$:

$$\tau_1(x, 0) = \tau_2(x, 0),\ \ \tau_1(0, y) = \tau_3(y, 0),\ \ \tau_2(0, z) = \tau_3(0, z)\ \ 0 \le x, y, z < \infty. \qquad (11)$$

**Theorem.** *[4] The Dirichlet problem for equation (3) in an unbounded domain $\Omega_3$ can have at most one solution.*

## Existence of the solution to the Dirichlet problem $D_3^3$

We will seek a solution to the Dirichlet problem (3) – (11) in the form

$$u\,(x, y, z) = \sum_{i=1}^{3} u_i(x, y, z),$$

where $u_1(x, y, z)$, $u_2(x, y, z)$, $u_3(x, y, z)$ are a solutions to the equation (3), satisfying the boundary conditions

$$u_1(x, y, 0) = \tau_1(x, y),\ \ u_1(x, 0, z) = 0,\ \ \ u_1(0, y, z) = 0, \qquad (12)$$

$$u_2(x,y,0)=0,\quad u_2(x,0,z)=\tau_2(x,z),\quad u_2(0,y,z)=0, \tag{13}$$

$$u_3(x,y,0)=0,\quad u_3(x,0,z)=0,\quad u_3(0,y,z)=\tau_3(y,z), \tag{14}$$

respectively.

**Lemma 1.** *If a function $\tau_1(x,y)$ can be represented by the formula (8), then the function*

$$u_1(x,y,z)=\sum_{j=1}^{4}(-1)^{j+1}u_{1j}(x,y,z) \tag{15}$$

*is a regular solution of the equation (3) in the domain $\Omega_3$, satisfying the boundary conditions (7) and (12), where*

$$u_{1j}(x,y,z)=\frac{z}{2\pi}\int_0^\infty\int_0^\infty\frac{\tau_1(t,s)}{\rho_{1j}^3}(1+\lambda\rho_{1j})e^{-\lambda\rho_{1j}}dtds,\quad j=1,2,3,4; \tag{16}$$

$$\rho_{11}=\sqrt{(t-x)^2+(s-y)^2+z^2},\quad \rho_{14}=\sqrt{(t-x)^2+(s+y)^2+z^2}, \tag{17}$$

$$\rho_{12}=\sqrt{(t+x)^2+(s-y)^2+z^2},\quad \rho_{13}=\sqrt{(t+x)^2+(s+y)^2+z^2}. \tag{18}$$

**Proof.** Now we will show that function $u_{11}(x,y,z)$ satisfies condition (4) of the Dirichlet problem.

By setting $t=x+z\mu,\ \ s=y+z\nu$, we transform formula (16) for $j=1$ to the form

$$u_{11}(x,y,z)=\frac{1}{2\pi}\int_0^\infty\int_0^\infty\frac{\tau_1(x+z\mu,y+z\nu)}{(1+\mu^2+\nu^2)^{3/2}}\left[1+\lambda z^2\left(1+\mu^2+\nu^2\right)\right]e^{-\lambda z^2\left(1+\mu^2+\nu^2\right)}d\mu d\nu \tag{19}$$

On the right-hand side of (19) we pass to the limit as $z\to 0$ and get

$$\lim_{z\to 0}u_{11}(x,y,z)=\tau_1(x,y)\,\frac{1}{2\pi}\int_0^\infty\int_0^\infty\frac{d\mu d\nu}{(1+\mu^2+\nu^2)^{3/2}}.$$

Using consistently the well-known identity [5, Eq. 4.623][1]

$$\int_0^\infty\int_0^\infty\varphi\left(a^2x^2+b^2y^2\right)dxdy=\frac{\pi}{4ab}\int_0^\infty\varphi(x)dx, \tag{20}$$

we obtain

$$\lim_{z\to 0}u_{11}(x,y,z)=\tau_1(x,y).$$

Likewise, everyone can easily get

$$\lim_{z\to 0}u_{1j}(x,y,z)=0,\quad j=2,3,4.$$

[1] There is a typo in this formula, the correct formula is (20).

So, the following equality is true for the function $u_1(x, y, z)$ defined in (15):

$$\lim_{z\to 0} u_1(x, y, z) = \tau_1(x, y).$$

Taking into account the obvious equalities

$$\rho_{11}|_{x=0} = \rho_{12}|_{x=0}, \ \ \rho_{11}|_{y=0} = \rho_{14}|_{y=0}, \ \ \rho_{12}|_{y=0} = \rho_{13}|_{y=0}, \ \ \rho_{13}|_{x=0} = \rho_{14}|_{x=0},$$

we establish that for there exist a following limits

$$\lim_{x\to 0} u_1(x, y, z) = 0, \ \ \lim_{y\to 0} u_1(x, y, z) = 0.$$

It is easy to see that as the point $(x, y, z)$ tends to infinity, i.e. $R \to \infty$, function (16) tends to zero. Then, by virtue of (8), we get

$$|u_{11}(x, y, z)| \le C_{11} z \int_0^\infty \int_0^\infty \frac{dtds}{(1 + t^2 + s^2)^{(1+\varepsilon_1)/2} \rho_{11}^2} \tag{21}$$

At the right-hand side of inequality (21), change the variables as follows: $t = R\mu$, $s = R\nu$. We obtain the inequalities

$$|u_{11}| \le \frac{C_{11}}{R^{\varepsilon_1}} \cdot \frac{z}{R} K(x, y, z; R)$$

where

$$K(x, y, z; R) = \int_0^\infty \int_0^\infty \frac{d\mu d\nu}{\left(\frac{1}{R^2} + \mu^2 + \nu^2\right)^{(1+\varepsilon_1)/2} \left[1 + \mu^2 + \nu^2 - \frac{2}{R}(\mu x + \nu y)\right]}.$$

Let us show that this proper double integral is bounded. Indeed, using the formula (20) and passing to the limit as $R \to \infty$, we calculate

$$\lim_{R\to\infty} K(x, y, z; R) = \frac{1}{4} \Gamma\left(\frac{1 + \varepsilon_1}{2}\right) \Gamma\left(\frac{2 + \varepsilon_1}{2}\right), \tag{22}$$

where $\Gamma(z)$ is a famous gamma function.

Using (22), we obtain

$$|u_{11}(x, y, z)| \le \frac{C}{R^{\varepsilon_1}}, \ \ \varepsilon_1 > 0, \tag{23}$$

where $C$ is a positive constant.

Now, it intermediately follows from estimate (23) that the function (16) vanishes at infinity. □

Similar statements are also true for functions $u_2(x, y, z)$ and $u_3(x, y, z)$ that satisfy the conditions (13) and (14), respectively, and vanish at infinity, where

$$u_i(x, y, z) = \sum_{j=1}^{4} (-1)^{j+1} u_{ij}(x, y, z), \ \ i = 2, 3,$$

$$u_{2j}(x,y,z) = \frac{y}{2\pi}\int_0^\infty\int_0^\infty \frac{\tau_2(t,s)}{\rho_{2j}^3}(1+\lambda\rho_{2j})e^{-\lambda\rho_{2j}}dtds, \tag{24}$$

$$\rho_{21} = \sqrt{(t-x)^2+y^2+(s-z)^2}, \quad \rho_{24} = \sqrt{(t-x)^2+y^2+(s+z)^2}, \tag{25}$$

$$\rho_{22} = \sqrt{(t+x)^2+y^2+(s-z)^2}, \quad \rho_{23} = \sqrt{(t+x)^2+y^2+(s+z)^2}, \tag{26}$$

$$u_{3j}(x,y,z) = \frac{x}{2\pi}\int_0^\infty\int_0^\infty \frac{\tau_3(t,s)}{\rho_{3j}^3}(1+\lambda\rho_{3j})e^{-\lambda\rho_{3j}}dtds, \tag{27}$$

$$\rho_{31} = \sqrt{x^2+(t-y)^2+(s-z)^2}, \quad \rho_{34} = \sqrt{x^2+(t-y)^2+(s+z)^2}, \tag{28}$$

$$\rho_{32} = \sqrt{x^2+(t+y)^2+(s-z)^2}, \quad \rho_{33} = \sqrt{x^2+(t+y)^2+(s+z)^2}. \tag{29}$$

Thus, we have proved the following

**Theorem 2.** *If a given functions $\tau_1$, $\tau_2$ and $\tau_3$ satisfy the conditions (8) – (11), then a function $u(x,y,z) = \sum_{i=1}^{3}\sum_{j=1}^{4}(-1)^{j+1}u_{ij}(x,y,z)$, where the functions $u_{ij}$ are defined in (16), (24) and (27), is a regular solution of equation (3) in the domain $\Omega_3$, satisfying the conditions (4) to (7).*

## Other problems for the Helmholtz equation in the first infinite octant

Let us consider the Helmholtz equation (3) in the infinite domain $\Omega_3$.

**The Neumann problem** $N_3^3$. Find a regular solution $u(x,y,z)$ to the Helmholtz equation (3) in the domain $\Omega_3$, satisfying the conditions

$$\lim_{z\to 0}\frac{\partial u(x,y,z)}{\partial z} = \nu_1(x,y), \quad 0 < x,y < \infty, \tag{30}$$

$$\lim_{y\to 0}\frac{\partial u(x,y,z)}{\partial y} = \nu_2(x,z), \quad 0 < x,z < \infty, \tag{31}$$

$$\lim_{x\to 0}\frac{\partial u(x,y,z)}{\partial x} = \nu_3(y,z), \quad 0 < y,z < \infty, \tag{32}$$

$$\lim_{R\to\infty} u(x,y,z) = 0, \;\; R = \sqrt{x^2+y^2+z^2}, \tag{33}$$

where $\nu_k(t,s)$ are given functions such that

$$\nu_1(x,y) = \frac{\tilde{\nu}_1(x,y)}{(1+x^2+y^2)^{\varepsilon_1/2}}, \;\; \tilde{\nu}_1(x,y) \in C(0<x,y<\infty), \;\; \varepsilon_1 > 0, \tag{34}$$

$$\nu_2(x,z) = \frac{\tilde{\nu}_2(x,z)}{(1+x^2+z^2)^{\varepsilon_2/2}}, \;\; \tilde{\nu}_2(x,z) \in C(0<x,z<\infty), \;\; \varepsilon_2 > 0, \tag{35}$$

$$\nu_3(y,z) = \frac{\tilde{\nu}_3(y,z)}{(1+y^2+z^2)^{\varepsilon_3/2}}, \;\; \tilde{\nu}_3(y,z) \in C(0<y,z<\infty), \;\; \varepsilon_3 > 0. \tag{36}$$

**Theorem 3.** *If a given functions* $\nu_1$, $\nu_2$ *and* $\nu_3$ *satisfy the conditions (34) – (36), then a function*

$$u(x,y,z) = -\frac{1}{2\pi}\sum_{i=1}^{3}\sum_{j=1}^{4}\int_0^\infty\int_0^\infty \frac{\nu_i(t,s)\,e^{-\lambda\rho_{ij}}}{\rho_{ij}}dtds, \ \ i=1,2,3; \ \ j=1,2,3,4$$

*is a regular solution of equation (3) in the domain* $\Omega_3$, *satisfying the boundary conditions (30) to (33), where* $\rho_{ij}$ *are defined in (17), (18), (25), (26), (28), (29).*

**The Dirichlet-Neuman problem** $D_3^2N_3^1$. Find a regular solution $u(x,y,z)$ to the Helmholtz equation (3) in the class of functions $C\left(\overline{\Omega_3}\right) \cup C^2(\Omega_3)$, satisfying the conditions

$$u(x,y,0) = \tau_1(x,y), \quad 0 \le x,y < \infty, \tag{37}$$

$$u(x,0,z) = \tau_2(x,z), \quad 0 \le x,z < \infty, \tag{38}$$

$$\lim_{x\to 0}\frac{\partial u(x,y,z)}{\partial x} = \nu_3(y,z), \quad 0 < y,z < \infty, \tag{39}$$

$$\lim_{R\to\infty} u(x,y,z) = 0, \ \ R = \sqrt{x^2+y^2+z^2}, \tag{40}$$

where $\tau_1(t,s)$,$\tau_2(t,s)$,$\nu_3(t,s)$ are given functions such that

$$\tau_1(x,y) = \frac{\tilde{\tau}_1(x,y)}{(1+x^2+y^2)^{(1+\varepsilon_1)/2}}, \ \ \tilde{\tau}_1(x,y) \in C(0 \le x,y < \infty), \ \varepsilon_1 > 0, \tag{41}$$

$$\tau_2(x,z) = \frac{\tilde{\tau}_2(x,z)}{(1+x^2+z^2)^{(1+\varepsilon_2)/2}}, \ \ \tilde{\tau}_2(x,z) \in C(0 \le x,z < \infty), \ \varepsilon_2 > 0, \tag{42}$$

$$\nu_3(y,z) = \frac{\tilde{\nu}_3(y,z)}{(1+y^2+z^2)^{\varepsilon_3/2}}, \ \ \tilde{\nu}_3(y,z) \in C(0 < y,z < \infty), \ \varepsilon_3 > 0. \tag{43}$$

In addition, the functions $\tau_1(x,y)$, $\tau_2(x,z)$ satisfy the concordance conditions in origin and in the lateral ribs of the domain $\Omega_3$:

$$\tau_1(x,0) = \tau_2(x,0), \ \ 0 \le x < \infty. \tag{44}$$

**Theorem 4.** *If a given functions* $\tau_1$, $\tau_2$ *and* $\nu_3$ *satisfy the conditions (41) – (44), then a function*

$$u(x,y,z) = \frac{z}{2\pi}\sum_{j=1}^{4}(-1)^{\left[\frac{j-1}{2}\right]}\int_0^\infty\int_0^\infty \frac{\tau_1(t,s)}{\rho_{1j}^3}(1+\lambda\rho_{1j})e^{-\lambda\rho_{1j}}dtds$$

$$+\frac{y}{2\pi}\sum_{j=1}^{4}(-1)^{\left[\frac{j-1}{2}\right]}\int_0^\infty\int_0^\infty \frac{\tau_2(t,s)}{\rho_{2j}^3}(1+\lambda\rho_{2j})e^{-\lambda\rho_{2j}}dtds$$

$$-\frac{1}{2\pi}\sum_{j=1}^{4}(-1)^{j+1}\int_0^\infty\int_0^\infty \frac{\nu_3(t,s)}{\rho_{3j}}e^{-\lambda\rho_{3j}}dtds,$$

*is a regular solution of equation (3) in the domain* $\Omega_3$*, satisfying the boundary conditions (37) to (40), where* $\rho_{ij}$ *are defined in (17),(18),(25),(26),(28),(29).*

Hereinafter, $[\alpha]$ denotes an integer part of the number $\alpha$.

**The Dirichlet-Neumann problem** $D_3^1N_3^2$. Find a regular solution $u(x,y,z)$ to the Helmholtz equation (3) in the domain $\Omega_3$, satisfying the conditions

$$\lim_{z\to 0}\frac{\partial u(x,y,z)}{\partial z}=\nu_1(x,y),\quad 0<x,y<\infty, \tag{45}$$

$$\lim_{y\to 0}\frac{\partial u(x,y,z)}{\partial y}=\nu_2(x,z),\quad 0<x,z<\infty, \tag{46}$$

$$u(0,y,z)=\tau_3(y,z),\quad 0\le y,z<\infty, \tag{47}$$

$$\lim_{R\to\infty}u(x,y,z)=0,\;\; R=\sqrt{x^2+y^2+z^2}, \tag{48}$$

where $\nu_1(t,s)$,$\nu_2(t,s)$,$\tau_3(t,s)$ are given functions such that

$$\nu_1(x,y)=\frac{\tilde{\nu}_1(x,y)}{(1+x^2+y^2)^{\varepsilon_1/2}},\;\; \tilde{\nu}_1(x,y)\in C(0<x,y<\infty),\; \varepsilon_1>0, \tag{49}$$

$$\nu_2(x,z)=\frac{\tilde{\nu}_2(x,z)}{(1+x^2+z^2)^{\varepsilon_2/2}},\;\; \tilde{\nu}_2(x,z)\in C(0<x,z<\infty),\; \varepsilon_2>0, \tag{50}$$

$$\tau_3(y,z)=\frac{\tilde{\tau}_3(y,z)}{(1+y^2+z^2)^{(1+\varepsilon_3)/2}},\;\; \tilde{\tau}_3(y,z)\in C(0\le y,z<\infty),\; \varepsilon_3>0. \tag{51}$$

**Theorem 5.** *If a given functions* $\nu_1$*,* $\nu_2$ *and* $\tau_3$ *satisfy the conditions (49) – (51), then a function*

$$u(x,y,z)=-\frac{1}{2\pi}\sum_{j=1}^{4}(-1)^{[\frac{j}{2}]}\int_0^\infty\int_0^\infty\frac{\nu_1(t,s)}{\rho_{1j}}e^{-\lambda\rho_{1j}}dtds$$

$$-\frac{1}{2\pi}\sum_{j=1}^{4}(-1)^{[\frac{j}{2}]}\int_0^\infty\int_0^\infty\frac{\nu_2(t,s)}{\rho_{2j}}e^{-\lambda\rho_{2j}}dtds+\frac{x}{2\pi}\sum_{j=1}^{4}\int_0^\infty\int_0^\infty\frac{\tau_3(t,s)}{\rho_{3j}^3}(1+\lambda\rho_{3j})e^{-\lambda\rho_{3j}}dtds$$

*is a regular solution of equation (3) in the domain* $\Omega_3$*, satisfying the conditions (45) to (48), where* $\rho_{ij}$ *are defined in (17),(18),(25),(26),(28),(29).*

## Boundary value problems for the Helmholtz equation in the first infinite square

*Let us consider the Helmholtz equation (3) in the infinite domain*

$$\Omega_2\equiv\{(x,y,z):-\infty<x<\infty,\;y>0,\;z>0\}.$$

***The Dirichlet problem*** $D_3^2$. *Find a regular solution* $u(x,y,z)$ *to the Helmholtz equation (3) in the domain* $\Omega_2$*, satisfying the conditions*

$$u(x,y,0)=\tau_1(x,y),\quad -\infty<x<\infty, 0\le y<\infty, \tag{52}$$

$$u(x,0,z)=\tau_2(x,z),\quad -\infty<x<\infty, 0\le z<\infty \tag{53}$$

$$\lim_{R\to\infty} u(x,y,z)=0,\ \ R=\sqrt{x^2+y^2+z^2}, \tag{54}$$

*where* $\tau_1(t,s)$ *and* $\tau_2(t,s)$ *are a given functions such that*

$$\tau_1(x,y)=\frac{\tilde{\tau}_1(x,y)}{(1+x^2+y^2)^{\varepsilon_1/2}},\ \tilde{\tau}_1(x,y)\in C(-\infty<x<\infty, 0\le y<\infty),\ \varepsilon_1>0, \tag{55}$$

$$\tau_2(x,z)=\frac{\tilde{\tau}_2(x,z)}{(1+x^2+z^2)^{\varepsilon_2/2}},\ \tilde{\tau}_2(x,z)\in C(-\infty<x<\infty, 0\le z<\infty),\ \varepsilon_2>0, \tag{56}$$

**Theorem 6.** *If a given functions* $\tau_1$ *and* $\tau_2$ *satisfy the conditions (55) and (56), then a function*

$u(x,y,z)=$

$$=\frac{z}{16\pi}\left[\int_0^\infty ds\int_{-\infty}^\infty \frac{\tau_1(t,s)(1+\lambda\rho_{11})e^{-\lambda\rho_{11}}}{\rho_{11}^3}dt-\int_0^\infty ds\int_{-\infty}^\infty \frac{\tau_1(t,s)(1+\lambda\rho_{14})e^{-\lambda\rho_{14}}}{\rho_{14}^3}dt\right]$$

$$+\frac{y}{16\pi}\left[\int_0^\infty ds\int_{-\infty}^\infty \frac{\tau_2(t,s)(1+\lambda\rho_{21})e^{-\lambda\rho_{21}}}{\rho_{21}^3}dt-\int_0^\infty ds\int_{-\infty}^\infty \frac{\tau_2(t,s)(1+\lambda\rho_{24})e^{-\lambda\rho_{24}}}{\rho_{24}^3}dt\right]$$

*is a regular solution of equation (3) in the domain* $\Omega_2$*, satisfying the conditions (52) to (54), where* $\rho_{ij}$ *are defined in (17) and (25).*

***The Dirichlet-Neuman problem*** $D_3^1N_3^1$*. Find a regular solution* $u(x,y,z)$ *to the Helmholtz equation (3) in the domain* $\Omega_2$ *, satisfying the conditions*

$$u(x,y,0)=\tau_1(x,y),\quad -\infty<x<\infty, 0\le y<\infty, \tag{57}$$

$$\lim_{y\to 0}\frac{\partial u(x,y,z)}{\partial y}=\nu_2(x,z),\quad -\infty<x<\infty, 0\le z<\infty, \tag{58}$$

$$\lim_{R\to\infty} u(x,y,z)=0,\ \ R=\sqrt{x^2+y^2+z^2}, \tag{59}$$

*where* $\tau_1(t,s)$, $\nu_2(t,s)$ *are given functions such that*

$$\tau_1(x,y)=\frac{\tilde{\tau}_1(x,y)}{(1+x^2+y^2)^{\varepsilon_1/2}},\ \tilde{\tau}_1(x,y)\in C(-\infty<x<+\infty, y<\infty),\ \varepsilon_1>0, \tag{60}$$

$$\nu_2(x,z)=\frac{\tilde{\nu}_2(x,z)}{(1+x^2+z^2)^{\varepsilon_2/2}},\ \tilde{\nu}_2(x,z)\in C(-\infty<x<+\infty, 0<z<\infty),\ \varepsilon_2>0, \tag{61}$$

**Theorem 7.** *If a given functions* $\tau_1$ *and* $\nu_2$ *satisfy the conditions (60) and (61), then a function*

$u(x,y,z)=$

$$=\frac{z}{16\pi}\left[\int_0^\infty ds\int_{-\infty}^\infty \frac{\tau_1(t,s)(1+\lambda\rho_{11})e^{-\lambda\rho_{11}}}{\rho_{11}^3}dt+\int_0^\infty ds\int_{-\infty}^\infty \frac{\tau_1(t,s)(1+\lambda\rho_{14})e^{-\lambda\rho_{14}}}{\rho_{14}^3}dt\right]$$

$$-\frac{1}{16\pi}\left[\int_0^\infty ds\int_{-\infty}^{\infty}\frac{\nu_2(t,s)\,e^{-\lambda\rho_{21}}}{\rho_{21}}dt-\int_0^\infty ds\int_{-\infty}^{\infty}\frac{\nu_2(t,s)\,e^{-\lambda\rho_{24}}}{\rho_{24}}dt\right]$$

*is a regular solution of equation (3) in the domain $\Omega_2$, satisfying the conditions (57) – (59), where $\rho_{ij}$ are defined in (17) and (25).*

*Let us consider the following Helmholtz equation (3) in the infinite domain $\Omega_2 \equiv \{(x,y,z): -\infty < x < \infty,\ y > 0,\ z > 0\}$.*

***The Neumann problem*** $N_3^2$. *Find a regular solution $u(x,y,z)$ to the Helmholtz equation (3) in the domain $\Omega_2$ , satisfying the conditions*

$$\lim_{z\to 0}\frac{\partial u(x,y,z)}{\partial z} = \nu_1(x,y), \quad -\infty < x < +\infty, 0 < y < \infty, \tag{62}$$

$$\lim_{y\to 0}\frac{\partial u(x,y,z)}{\partial y} = \nu_2(x,z), \quad -\infty < x < +\infty, 0 < z < \infty, \tag{63}$$

$$\lim_{R\to\infty} u(x,y,z) = 0, \ \ R = \sqrt{x^2+y^2+z^2}, \tag{64}$$

*where $\nu_k(t,s)$ are given functions such that*

$$\nu_1(x,y) = \frac{\tilde{\nu}_1(x,y)}{(1+x^2+y^2)^{\varepsilon_1/2}},\ \tilde{\nu}_1(x,y) \in C(-\infty < x < +\infty, 0 < y < \infty),\ \varepsilon_1 > 0, \tag{65}$$

$$\nu_2(x,z) = \frac{\tilde{\nu}_2(x,z)}{(1+x^2+z^2)^{\varepsilon_2/2}},\ \tilde{\nu}_2(x,z) \in C(-\infty < x < +\infty, 0 < z < \infty),\ \varepsilon_2 > 0, \tag{66}$$

**Theorem 8.** *If a given functions $\nu_1$ and $\nu_2$ satisfy the conditions (65) and (66), then a function*

$$u(x,y,z) = -\frac{1}{16\pi}\left[\int_0^\infty ds\int_{-\infty}^{\infty}\frac{\nu_1(t,s)\,e^{-\lambda\rho_{11}}}{\rho_{11}}dt+\int_0^\infty ds\int_{-\infty}^{\infty}\frac{\nu_1(t,s)\,e^{-\lambda\rho_{14}}}{\rho_{14}}dt\right]$$

$$-\frac{1}{16\pi}\left[\int_0^\infty ds\int_{-\infty}^{\infty}\frac{\nu_2(t,s)\,e^{-\lambda\rho_{21}}}{\rho_{21}}dt+\int_0^\infty ds\int_{-\infty}^{\infty}\frac{\nu_2(t,s)\,e^{-\lambda\rho_{24}}}{\rho_{24}}dt\right]$$

*is a regular solution of equation (3) in the domain $\Omega_2$, satisfying the conditions (62) – (64), where $\rho_{ij}$ are defined in (17) and (25).*

## Boundary value problems for the Helmholtz equation in the half-space

Let us consider the following Helmholtz equation (3) in the infinite domain $\Omega_1 \equiv \{(x,y,z): -\infty < x,y < \infty, \ \ z > 0\}$

**The Dirichlet problem** $D_3^1$. Find a regular solution $u(x,y,z)$ to the Helmholtz equation (3) in the domain $\Omega_1$ , satisfying the conditions

$$u(x,y,0) = \tau_1(x,y), \quad -\infty < x,y < \infty, \tag{67}$$

$$\lim_{R\to\infty} u(x,y,z) = 0, \quad R = \sqrt{x^2+y^2+z^2}, \tag{68}$$

where $\tau_1(t,s)$ are given functions such that

$$\tau_1(x,y) = \frac{\tilde{\tau}_1(x,y)}{(1+x^2+y^2)^{\varepsilon_1/2}}, \ \tilde{\tau}_1(x,y) \in C(-\infty < x, y < +\infty, ), \ \varepsilon_1 > 0, \tag{69}$$

Note, that the Dirichlet problem for Helmholtz equation (3) in case of half-space is found in the books for students [16, 17].

**Theorem 9.** *If a given function $\tau_1$ satisfies the condition (69), then a function*

$$u(x,y,z) = \frac{z}{2\pi} \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} \frac{\tau_1(t,s)(1+\lambda\rho_{11})e^{-\lambda\rho_{11}}}{\rho_{11}^3} dt \tag{70}$$

*is a regular solution of equation (3) in the domain $\Omega_1$, satisfying the conditions (67) and (68), where $\rho_{11}$ is defined in (17).*

**The Neumann problem** $N_3^1$. Find a regular solution $u(x,y,z)$ to the Helmholtz equation (3) in the domain $\Omega_1$ , satisfying the conditions

$$\lim_{z\to 0} \frac{\partial u(x,y,z)}{\partial z} = \nu_1(x,y), \quad -\infty < x < +\infty, -\infty < y < +\infty, \tag{71}$$

$$\lim_{R\to\infty} u(x,y,z) = 0, \quad R = \sqrt{x^2+y^2+z^2}, \tag{72}$$

where $\nu_1(t,s)$ is a given function such that

$$\nu_1(x,y) = \frac{\tilde{\nu}_1(x,y)}{(1+x^2+y^2)^{\varepsilon_1/2}}, \ \tilde{\nu}_1(x,y) \in C(-\infty < x < +\infty, -\infty < y < +\infty), \ \varepsilon_1 > 0, \tag{73}$$

Note, that the Neumann problem for Helmholtz equation (3) in case of half-space is found in [16, 17].

**Theorem 10.** *If a given function $\nu_1$ satisfies the condition (73), then a function*

$$u(x,y,z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} \frac{\nu_1(t,s)\, e^{-\lambda\rho_{11}}}{\rho_{11}} dt \tag{74}$$

*is a regular solution of equation (3) in the domain $\Omega_1$, satisfying the boundary conditions (71) and (72), where $\rho_{11}$ is defined in (17).*

**Remark 1**. The solutions (70) and (74) of the Dirichlet and Neumann problems are found in [17].

**Remark 2**. Uniqueness of solutions of the problems posed is proved using the extremum principle for elliptic equations.

## Conclusion

In this paper we presented solutions of 9 boundary value problems for the three-dimensional Helmholtz equation in infinite domains in explicit and convenient forms for further research. The results of this paper can be useful in the study of boundary value problems for a three-dimensional equation of mixed type, because from the explicit solutions of the posed problems it is easy to derive functional relationships between the traces of the desired solution and its derivative along the normal, brought to the plane of change of the equation type.

## References


1. Hu G., Rathsfeld A. Radiation conditions for the Helmholtz equation in a half plane filled by inhomogeneous periodic material, *Journal of Differential Equations*, 2024. vol. 388, pp. 215–252.
2. Chandler-Wilde S. N. Boundary value problems for the Helmholtz equation in a half-plane, *Conference: Third International Conference on Mathematical and Numerical Aspects of Wave Propagation Phenomena*, 1994. vol. 94, pp. 1–12.
3. Frank-Olme S., Ernst S. Boundary Value Problems for the Helmholtz Equation in an Octant, *Integral Equations and Operator Theory*, 2008. vol. 62, pp. 269–300.
4. Sveshnikov A. G., Bogolyubov A. N., Kravtsov V. V. *Lectures on Mathematical Physics*. Moscow: Moscow State University, 1993. 352 pp. (in Russian)
5. Gradshteyn I. S., Ryzhik I. M. *Table of integrals, series and products*. Amsterdam: Academic Press, 2007. 1172 pp.
6. Ergashev T. G., Safarbaeva N. M. Holmgren's problem for a multidimensional Helmholtz equation with one singular coefficient, *Bulletin of the Institute of Mathematics*, 2020. vol. 1, pp. 127 – 135.
7. Ergashev T. G., Hasanov A. Holmgren problem for elliptic equation with singular coefficients, *Vestnik KRAUNC. Fiz.-mat. nauki.*, 2020. vol. 32, no. 3, pp. 159 – 175.
8. Ergashev T. G., Hasanov A. Fundamental solutions of the bi-axially symmetric Helmholtz equation, *Uzbek Mathematical Journal*, 2018. vol. 1, pp. 55 – 64.
9. Ergashev T. G. Third double-layer potential for a generalized bi-axially symmetric Helmholtz equation, *Ufa Mathematical Journal*, 2018. vol. 10, pp. 111 – 121.
10. Repin O. A., Lerner M. E. On the Dirichlet problem for the generalized bioxially symmetric Helmholtz equation in the first quadrant, *Vestnik Samarsk. Gos. Tekh. Universiteta, Ser. fiz.-matem. nauki*, 1998. vol. 6, pp. 5 – 8 (in Russian).
11. Samko S.G., Kilbas A.A., Marichev O.I. *Fractional integrals and derivatives. Theory and applications*. Amsterdam: Gordon and Breach Science Publishers, 1993. 976 pp.
12. Juraev D. A., Agarwal P., Elsayed E. E., Targyn N. Applications of the Helmholtz equation, *Advanced Engineering Days*, 2023. vol. 8, pp. 28 – 30.
13. Juraev D. A. On the Cauchy problem for matrix factorizations of the Helmholtz equation in a bounded domain, *Siberian Electronic Mathematical Reports*, 2018. vol. 15, pp. 11 – 20.
14. Juraev D. A. The construction of the fundamental solution of the Helmholtz equation, *Reports of the Academy of Sciences of the Republic of Uzbekistan*, 2012. vol. 2, pp. 14 – 17.
15. Zwillinger D. *Handbook of Differential Equations*. Boston, MA: Academic Press, 1997. 129 pp.
16. Tikhonov, A. N. and Samarskii, A. A. *Equations of Mathematical Physics, Dover Publ.* New York: Dover Publ, 1990. 829 pp.
17. Budak, B. M., Samarskii, A. A., and Tikhonov, A. N. *Collection of Problems on Mathematical Physics*. Moscow: Nauka, 1980. 829 pp. (in Russian)
18. Kapilevich M.B. On one equation of mixed elliptic-hyperbolic type., *Matematicheskii sbornik*, 1952. vol. 30(72), pp. 11 – 38 (in Russian).
19. Marichev O.I. Integral representation of solutions of the generalized biaxially symmetric Helmholtz equation and its inversion formula., *Differensialniye uravneniya*, 1978. vol. 14, pp. 1824 – 1831 (in Russian).


### Information about the author

*Arzikulov Zafarjon Odilovich* ✉ – Doktorant, Fac. of Phys. & Department of Higher Mathematics, Fergana Politechnical Institute Fergana, Uzbekistan, ORCID 0009-0004-2965-4566.