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Optimal eight-order three-step iterative methods for solving systems of nonlinear equations

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Abstract. In this paper, we for the first time propose the extension of optimal eighth-order methods to multidimensional case. It is shown that these extensions maintained the optimality properties of the original methods. The computational efficiency of the proposed methods is compared with that of known methods. Numerical experiments are included to confirm the theoretical results and to demonstrate the efficiency of the methods.

Key words and phrases: newton-type methods, systems of nonlinear equations, convergence order, optimality and extension of methods, efficiency index

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1. Introduction

The problem of finding solution of nonlinear system

$$F(x) = 0, \;\; x = (x_1, x_2, \cdots, x_n)^T \in R^n, \tag{1}$$

is often appeared in different fields of science and engineering (see [1–13] and references therein). In general, the solutions of systems cannot be obtained exactly; therefore, numerous iterative methods have been developed and are widely used to solve Equation (1). Quite recently, several methods with vector or scalar coefficients have appeared in the literature [4, 5, 7, 11]. They are distinguished by the simplicity of the algorithm compared to other well-known methods with matrix coefficients. Methods with vector coefficients constructed in R^n with point-wise multiplication and division of vectors [5, 12]. Let $x = (x_1, x_2, \dots, x_n)^T \in R^n$, $y = (y_1, y_2, \dots, y_n)^T \in R^n$. Then

$$x \cdot y = (x_1 y_1, x_2 y_2, \dots, x_n y_n)^T \in R^n, \tag{2}$$

$$\frac{x}{y} = \left(\frac{x_1}{y_1}, \frac{x_2}{y_2}, \dots, \frac{x_n}{y_n}\right)^T \in R^n. \tag{3}$$

A direct consequence of (2), (3) is

$$x \cdot y = y \cdot x, \tag{4}$$

$$x^2 = x \cdot x = (x_1^2, x_2^2, \dots, x_n^2)^T, \tag{5}$$

$$\mathbf{1} = (1, 1, \dots, 1)^T. \tag{6}$$

Using (2), (3), (4), (5) and (6) it is easy to show that the following vector expansion holds true

$$\frac{\mathbf{1}}{\mathbf{1} - x} = \mathbf{1} + x + x^2 + x^3 \dots, \quad \text{for } \|x\| = \max_i |x_i| < 1. \tag{7}$$

We consider three-step iterations

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1}F(x_k), \\ z_k &= \phi_p(x_k, y_k), \qquad \big(z_k = y_k - \bar{\tau}_k F'(x_k)^{-1}F(y_k)\big), \\ x_{k+1} &= z_k - \alpha_k F'(x_k)^{-1}F(z_k). \end{aligned} \tag{8}$$

Here $z_k = \phi_p(x_k, y_k)$ is an iteration of order $p \geqslant 2$ i.e., $F(z_k) = O(F(x_k)^p)$. The following Theorem 1 was proved in [8, 9].

Theorem 1. *Suppose that $F(x)$ is sufficiently differentiable in the open convex domain $D \in R$ containing the simple zero x^* of* (1) *and $F(x)$ is continuous and non-singular at x^*. Let $x_0 \in D$ be an initial approximation sufficiently close to x^*. Then the local order of convergence of iteration* (8) *equal to $p + 4$ if and only if α_k satisfies*

$$\alpha_k = I + 2\eta_k + 6\eta_k^2 + 3d_k + 20\eta_k^3 + 20d_k\eta_k + c_k + O(h^4), \tag{9}$$

where

$$\eta_k = \frac{1}{2}F'(x_k)^{-1}F''(x_k)\xi_k, \quad d_k = -\frac{1}{6}F'(x_k)^{-1}F'''(x_k)\xi_k^2,$$
$$c_k = \frac{1}{6}F'(x_k)^{-1}F^{(4)}(x_k)\xi_k^3, \quad \xi_k = F'(x_k)^{-1}F(x_k).$$

It should be pointed out that in the construction of high-order iterations, in particular, in the proof of the above mentioned theorem, the following Taylor expansion lemma and permutation properties [4] of q-derivative of the vector function $F(x)$ were used.

Lemma 1. *Let $F : D \subseteq R^n \to R^n$ be $p-$times Fréchet differentiable in a open convex set $D \subseteq R^n$, then for any $x, \hat{h} \in D$ the following expression holds:*

$$F(x + \hat{h}) = F(x) + F'(x)\hat{h} + \frac{1}{2!}F''(x)\hat{h}^2 + \frac{1}{3!}F'''(x)\hat{h}^3 + \dots + \frac{1}{p!}F^{(p)}(x)\hat{h}^p + R_p,$$

where

$$\|R_p\| \le \frac{1}{p!}\sup_{0<t<1}\|F^{(p)}(x + t\hat{h})\|\|\hat{h}\|^p \ \text{ and } \ \hat{h}^p = (\overbrace{\hat{h}, \hat{h}, \cdots, \hat{h}}^{p}),$$

and $\|\cdot\|$ denotes any norm in R^n, or a corresponding operator norm.

The q-th derivative of F at $u \in R^n$, $q \ge 1$, is the q-linear function $F^{(q)}(u) : R^n \times \dots \times R^n \to R^n$ such that $F^{(q)}(u)\left(v_1, \dots, v_q\right) \in R^n$

(i) $F^{(q)}(u)\left(v_1, \dots, v_{q-1}, \cdot\right) \in L\left(R^n\right)$,
(ii) $F^{(q)}(u)\left(v_{\sigma(1)}, \dots, v_{\sigma(q)}\right) = F^{(q)}(u)\left(v_1, \dots, v_q\right)$, for all permutations σ of $\{1, 2, \dots, q\}$.

From the above properties, we can use the following notation:

(a) $F^{(q)}(u)\left(v_1, \dots, v_q\right) = F^{(q)}(u)v_1 \dots v_q$,
(b) $F^{(q)}(u)v^{q-1}F^{(p)}(u)v^p = F^{(q)}(u)F^{(p)}(u)v^{q+p-1}$.

For convenience, we also recall the conjecture given by [4].

Conjecture. The order of convergence of any iterative method without memory for solving nonlinear systems cannot exceed $2^{k_1+k_2-1}$, where k_1 is the number of evaluation of the Jacobian matrix and k_2 is the number of evaluations of the nonlinear function per iteration, and $k_1 \leq k_2$. When the scheme reach this upper bound, we say that it is optimal.

In the second step of iteration (8) we use

$$z_k = y_k - \bar{\tau}_k F'(x_k)^{-1} F(y_k),$$

with vector parameter

$$\bar{\tau}_k = \frac{\mathbf{1} + a\Theta_k + b\Theta_k^2}{\mathbf{1} + (a-2)\Theta_k + d\Theta_k^2} = \mathbf{1} + 2\Theta_k + O(F(x_k)^2), \quad a, b, d \in R, \tag{10}$$

where

$$\Theta_k = \frac{F(y_k)}{F(x_k)}. \tag{11}$$

In [12], it was proven that $p = 4$ under choice (10). Using $\Theta_k = O(F(x_k))$ and the expansion (7), we rewrite (10) as

$$\begin{aligned} \bar{\tau}_k = \mathbf{1} &+ 2\Theta_k + (2(2-a) + b - d)\,\Theta_k^2 + (2(2-a)^2 \\ &+ (2-a)(b-2d) - ad)\Theta_k^3 + \dots . \end{aligned} \tag{12}$$

The optimal fourth-order two-step methods were first proposed in [4, 12]. The purpose of this paper is to develop families of optimal eighth-order methods based on the optimal fourth-order methods introduced in [12].

2. Extensions of several iterations to multidimensional case

First, we consider three-step iteration

$$\begin{aligned} y_k &= x_k - \frac{f(x_k)}{f'(x_k)}, \\ z_k &= y_k - \bar{\tau}_k \frac{f(y_k)}{f'(x_k)}, \\ x_{k+1} &= z_k - \alpha_k \frac{f(z_k)}{f'(x_k)}, \end{aligned} \tag{13}$$

for solving nonlinear scalar equation $f(x) = 0$. Let the iteration parameter $\bar{\tau}_k$ satisfies

$$\bar{\tau}_k = 1 + 2\theta_k + \beta\theta_k^2 + \gamma\theta_k^3 + O(f_k^4), \quad \theta_k = \frac{f(y_k)}{f'(x_k)}. \tag{14}$$

In [8], it was proven that the local order of the iteration (13) is eight if and only if α_k satisfies the condition

$$\alpha_k = 1 + 2\theta_k + (\beta + 1)\theta_k^2 + (2\beta + \gamma - 4)\theta_k^3 + (1 + 4\theta_k)\frac{f(z_k)}{f(y_k)} + O(f_k^4). \quad (15)$$

We now consider a formal extension of the iteration (13) to R^n with point-wise multiplication and division of vectors replacing $f(x)$ by $F(x)$. Then (13) leads to (8). The parameter choices (14) and (15) lead to

$$\bar{\tau}_k = \mathbf{1} + 2\Theta_k + \beta\Theta_k^2 + \gamma\Theta_k^3 + \dots, \quad (16)$$

and

$$\alpha_k = \mathbf{1} + 2\Theta_k + (\beta + 1)\Theta_k^2 + (2\beta + \gamma - 4)\Theta_k^3 + (\mathbf{1} + 4\Theta_k)\frac{F(z_k)}{F(y_k)}, \quad (17)$$

respectively, where Θ_k is defined by formula (11). In this connection, the question arises: how is the local order of iteration (8) preserved or reduced when vector parameters satisfy conditions (16) and (17)? The formal extension with parameters satisfying (16) and (17) seems to be actual extension with good convergence properties. Namely, we can state and prove the following result:

Theorem 2. *Suppose that the assumptions of Theorem 1 are satisfied. Then the iterative method* (8) *has a eight-order of convergence if and only if the parameters $\bar{\tau}_k$ and α_k satisfy* (16) *and* (17)*, respectively.*

Proof. From (10), (12) and (16) it is clear that $p = 4$ maintained under (16). To prove the theorem it suffices to show what the conditions (9) and (17) are equivalent. Using Lemma 1 and permutations properties (i), (ii) one can easily obtain

$$\Theta_k = \frac{F(y_k)}{F(x_k)} = \eta_k + d_k + \frac{c_k}{4} + O\big(F(x_k)^4\big), \quad (18)$$

$$s_k = \frac{F(z_k)}{F(y_k)} = I - \hat{t}_k\bar{\tau}_k + \bar{\tau}_k^2\eta_k(\eta_k + 4d_k) + O\big(F(x_k)^4\big), \quad (19)$$

where

$$\hat{t}_k = F'(x_k)^{-1}F'(y_k) = I - 2\eta_k - 3d_k - c_k + O\big(F(x_k)^4\big). \quad (20)$$

Substituting (16), (20) into (19) we obtain

$$s_k = d_k + (5 - \beta)\eta_k^2 + \frac{c_k}{2} + (14 - 2\beta)\eta_k d_k + (4 + 2\beta - \gamma)\eta_k^3 + O\big(F(x_k)^4\big). \quad (21)$$

Further using (18), (21) in (17) we obtain (9) i.e., from (17) we get (9). Analogously, if we take into account formulas (18) and (21), then (9) immediately leads to (17). □

Remark 1. If we ignore terms with order $O\big(F(x_k)^3\big)$ in (17) then the order of convergence of iteration (8) decreased by unit i.e., the iteration (8) has $p + 3$ order of convergence if and only if $\bar{\tau}_k$ and α_k satisfy (16) and

$$\alpha_k = \mathbf{1} + 2\Theta_k + (\beta + 1)\Theta_k^2 + \frac{F(z_k)}{F(y_k)} + O\big(F(x_k)^3\big),$$

respectively.

Since $p = 4$ under choice (16), the condition (17) guarantees eight-order convergence of iteration (8). This extension (8) with (16), (17) is scheme with vector coefficients and optimal according to the conjecture, because of $k_1 = 1$, $k_2 = 3$. In [2] the authors found that it is better not to use polynomials as weight functions. Following this idea, we use (10) and rewrite formula (17) in rational function form as

$$\alpha_k = \left(\frac{\mathbf{1} - \Theta_k}{\mathbf{1} - 2\Theta_k}\right)^2 H_k - \frac{\gamma_1 \Theta_k^2}{(\mathbf{1} - \gamma_2 \Theta_k)}, \tag{22}$$

where

$$t_k = \frac{F(z_k)}{F(x_k)}, \quad s_k = \frac{F(z_k)}{F(y_k)}, \quad H_k = \frac{\mathbf{1} + \dfrac{t_k}{(\mathbf{1} + \alpha t_k)}}{(\mathbf{1} - t_k)(\mathbf{1} - s_k)},$$
$$\gamma_1 = 2a + d - b, \quad \gamma_2 = \frac{8 + 2\gamma_1 - 2(2-a)^2 - (2-a)(b-2d) + ad}{\gamma_1}.$$

So, we obtain a family of optimal eight-order iterations

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1} F(x_k), \\ z_k &= y_k - \bar{\tau}_k F'(x_k)^{-1} F(y_k), \\ x_{k+1} &= z_k - \alpha_k F'(x_k)^{-1} F(z_k), \end{aligned} \tag{23}$$

where $\bar{\tau}_k$ and α_k are given by (10) and (22) respectively. We consider a simpler case of (23). Let $b = d = 0$. Then, equation (23) simplifies to:

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1} F(x_k), \\ z_k &= y_k - \frac{\mathbf{1} + a\Theta_k}{\mathbf{1} + (a-2)\Theta_k} F'(x_k)^{-1} F(y_k), \\ x_{k+1} &= z_k - \left(\left(\frac{\mathbf{1} - \Theta_k}{\mathbf{1} - 2\Theta_k}\right)^2 H_k - \frac{2a\Theta_k^2}{(\mathbf{1} + (a-6)\Theta_k)}\right) F'(x_k)^{-1} F(z_k), \quad a \in R. \end{aligned} \tag{24}$$

In particular, when $a = 0$ the iteration (24) leads to one, which are called the extension of Sharma's method [6]. In [2], Chun and Neta conducted a comparative analysis of well-known eighth-order optimal methods based on the average number of iterations and CPU time. They found that WL [13], Sharma [6] and CL [3] methods are best one. In principle, one can formulate the extension of any eight-order method to a multidimensional case replacing $f(x)$, $f(y)$, $f(z)$, $f[u_k, v_k]$ by $F(x)$, $F(y)$, $F(z)$, $F[u_k, v_k]$ respectively.

In construction of extension often used formulas

$$F[u_k, v_k] = \frac{F(v_k) - F(u_k)}{v_k - u_k}, \tag{25}$$

and

$$F[x_k, y_k] = F'(x_k)(\mathbf{1} - \Theta_k), \tag{26}$$

$$F[y_k, z_k] = F'(x_k)\frac{\mathbf{1} - s_k}{\bar{\tau}_k}, \tag{27}$$

$$F[x_k, z_k] = F'(x_k)\frac{\mathbf{1} - t_k}{\mathbf{1} + \bar{\tau}_k \Theta_k}, \quad t_k = \frac{F(z_k)}{F(x_k)} = \Theta_k s_k, \tag{28}$$

that followed by (8). So, we restrict here only extensions of the WL [13], SS [6], CL [3], BRW [13] methods. First we note that when $a = b = d = 0$ the iteration (23) includes an extension of Sharma et al. [6] to multidimensional case as a particular case.

1. Now we consider the iteration (8) with parameters $\bar{\tau}_k$, α_k given by (10) and

$$\alpha_k = \frac{F'(x_k)}{F[y_k, z_k] + 2\left(F[x_k, z_k] - F[y_k, x_k]\right) + \dfrac{y_k - z_k}{y_k - x_k}\left(F[y_k, x_k] - F'(x_k)\right)}. \tag{29}$$

Using formulas (25), (26), (27) and (28) and expansion (7) one can rewrite (29) as

$$\alpha_k = \bar{\tau}_k\left(\mathbf{1} + \Theta_k^2 + (2\beta - 6)\Theta_k^3 + (\mathbf{1} + 2\Theta_k)s_k\right) + O\big(F(x_k)^4\big). \tag{30}$$

If we take into account (16), then (30) leads to (17). Then by Theorem 2 the iteration (8) with parameter $\bar{\tau}_k$, α_k given by (10) and (29) is optimal eight-order convergence. Thus, we obtain a wide class of optimal eight-order methods.

Note that the formula (29) is the extension formula (3.63) given by [9, 10] to multidimensional case. When $a = b = d = 0$ in (10), $\bar{\tau}_k$ becomes as

$$\bar{\tau}_k = \frac{\mathbf{1}}{\mathbf{1} - 2\Theta_k}. \tag{31}$$

The iteration (8) with $\bar{\tau}_k$ and α_k given by (31) and (29) is extension of WL method [13]. In [10] was given extension of some classes of optimal three-point iterations for solving nonlinear equations, that works well for any choice of parameter $\bar{\tau}_k$ satisfying the condition (14). According to Theorem 2, these extensions suggested in [10] are immediately extended to multidimensional case.

2. For example, we consider the iteration (8) parameters $\bar{\tau}_k$ satisfying (10) and α_k given by

$$\alpha_k = \left(\mathbf{1} + 2t_k - 2(2a+1)\Theta_k^3\right)\frac{F'(x_k)}{F[z_k, y_k] + \dfrac{z_k - y_k}{z_k - x_k}\left(F[z_k, x_k] - F'(x_k)\right)}. \tag{32}$$

As before, it is easy to show that α_k given by (32) satisfies the condition (17). So by Theorem 2 the iteration (8) with parameters given by (10), (32) has a eight-order convergence. Note that a formula (32) is a multidimensional extension of optimal modification (23) in [10]. When $a = -\frac{1}{2}$ the iteration(8) with parameters $\bar{\tau}_k$, α_k given by (10), (32) is considered as multidimensional extension of BRW method [1]. Our finding and generalization show that, in particular, BRW [1] is optimal not only for $a = -\frac{1}{2}$, but also it is optimal for any a. So we call the methods (8) with (10), (32) multidimensional extension of the optimal modification of BRW methods.

3. Now we consider the iteration (8) with parameters $\bar{\tau}_k$ satisfying the condition (16) and α_k given by

$$\alpha_k = \frac{F'(x_k)\left[\mathbf{1} + A\Theta_k + B\Theta_k^2 + C\Theta_k^3 + (\delta + \Delta\Theta_k)s_k\right]}{\omega_1 F[x_k, z_k] + \omega_2 F[z_k, y_k] + \omega_3 F[x_k, y_k]}, \tag{33}$$

where $\omega_1 + \omega_2 + \omega_3 = 1$ and A, B, C, δ, Δ are free parameters to be determined properly. As before, it is easy to show that the iteration (8) with $\bar{\tau}_k$, α_k given by (10) and (33) is optimal eight-order convergence when

$$A = \delta = 1 - \omega_2, \quad B = (\tilde{\beta} - 2)(1 - \omega_2) + 1 - \omega_1, \quad \Delta = 3 - \omega_1 - \omega_2$$
$$C = \tilde{\gamma}(1 - \omega_2) + \tilde{\beta}(1 + \omega_2 - \omega_1) + \omega_1 - \omega_2 - 5.$$

Table 1

Class of optimal eight-order iterations (33)

ω_1	ω_2	ω_3
0	1	0
1	0	0
0	0	1
-1	2	0
1	1	-1
-1	1	1

Thus, we have a wide class of optimal eight-order iterations (8), (10) and (33) containing five free parameters a, b, d and ω_1, ω_2. The formula (33) is an extension of formula (11) in the paper [10] to the multidimensional case. The interesting and easy cases of (33) are in Table 1.

This family of methods (8), (10), (33) contains many multidimensional extensions of well-known eight-order methods for solving nonlinear equations (See [9, 10] and references therein).

4. Now we will derive an extension of CL method [3]. To do this we consider the weight function α_k as

$$\alpha_k = \frac{\mathbf{1}}{\left(\mathbf{1} - H(\Theta_k) - J(t_k) - P(s_k)\right)^2}, \tag{34}$$

where the weight functions should satisfy the following conditions to guarantee eight order:

$$H(0) = 0, \quad H'(0) = 1, \quad H''(0) = \beta - 2, \quad H'''(0) = 3(\gamma - \beta - 2), \tag{35}$$

$$J(0) = 0, \quad J'(0) = 1/2, \quad P(0) = 0, \quad P'(0) = 1/2, \tag{36}$$

where β and γ are constants in (16). These functions satisfying the conditions (35), (36) are given by

$$J(t_k) = \frac{1}{2}\frac{t_k}{\mathbf{1} + \delta_1 t_k}, \quad P(s_k) = \frac{1}{2}\frac{s_k}{\mathbf{1} + \delta_2 s_k}, \quad H(\Theta_k) = \frac{\Theta_k}{2}\frac{2 + (3 - 4\delta_3)\Theta_k}{1 + (1 - 2\delta_3)\Theta_k + \delta_3\Theta_k^2},$$

where δ_1, δ_2, $\delta_3 \in R$.

If we choose $\bar{\tau}_k$ as

$$\bar{\tau}_k = \frac{\mathbf{1}}{(1 - \Theta_k)^2}, \quad (\beta = 3,\ \gamma = 4 \text{ in } (16)), \tag{37}$$

then the iteration (8) with $\bar{\tau}_k$, α_k given by (37) and (34) is extension of CL methods [3].

3. Transition to schemes with scalar coefficients

The eighth-order family of iterations (8) with parameters $\bar{\tau}_k$ and α_k satisfying condition (16) and (17) is scheme with vector coefficients. Now we will show that it is possible transition from scheme with vector coefficients to scheme with scalar coefficients. Indeed, using (16) one can obtain

$$\bar{\tau}_k F(y_k) = (\mathbf{1} + \beta\Theta_k^2)F(y_k) + (2\Theta_k^2 + \gamma\Theta_k^4)F(x_k). \tag{38}$$

Analogously, using (17) and the following

$$t_k = \frac{F(z_k)}{F(x_k)}, \quad s_k = \frac{F(z_k)}{F(y_k)}, \quad s_k\Theta_k = \frac{F(z_k)}{F(x_k)}$$

we get

$$\alpha_k F(z_k) = F(z_k) + (s_k^2 + (\beta+1)\Theta_k t_k)F(y_k) + \big(2\Theta_k^2 t_k + (2\beta+\gamma-4)\Theta_k^2 t_k + 4s_k^2\Theta_k^2\big)F(x_k). \tag{39}$$

Using transition rules [11] (replacing the pointwise multiplication by dotted product)

$$\begin{gathered}\Theta_k t_k = \frac{F(z_k)F(y_k)}{F(x_k)^2} \Longleftrightarrow \alpha_k = \frac{(F(z_k), F(y_k))}{\|F(x_k)\|^2},\\ s_k^2 = \left(\frac{F(z_k)}{F(y_k)}\right)^2 \Longleftrightarrow \beta_k = \frac{\|F(z_k)\|^2}{\|F(y_k)\|^2}, \quad \Theta_k^2 \Longleftrightarrow \upsilon_k = \frac{\|F(y_k)\|^2}{\|F(x_k)\|^2},\\ s_k^2\Theta_k^2 \Longleftrightarrow \delta_k = \upsilon_k\beta_k = \frac{\|F(z_k)\|^2}{\|F(x_k)\|^2}, \quad \mathbf{1} \Longleftrightarrow 1,\end{gathered} \tag{40}$$

in (38) and (39) the iterations (8) can be rewritten as follows:

$$\begin{gathered}y_k = x_k - F'(x_k)^{-1}F(x_k),\\ z_k = y_k - F'(x_k)^{-1}\big((1+\beta\upsilon_k)F(y_k) + (2\upsilon_k + \gamma\upsilon_k^2)F(x_k)\big),\\ x_{k+1} = z_k - F'(x_k)^{-1}\big(F(z_k) + (\beta_k + (\beta+1)\alpha_k)F(y_k) + (2\alpha_k + (2\beta+\gamma-4)\upsilon_k\alpha_k + 4\delta_k)F(x_k)\big).\end{gathered} \tag{41}$$

Thus, we obtain first a family of optimal eighth-order iterations (41) with scalar coefficients. The conversation of iterations (41) to (8) with vector coefficients (38) and (39) is obvious by virtue of rules (40). When $\beta = 1$ and $\gamma = 0$, the iteration (41) reduces to the method obtained by Cordero et al. in [14] as a particular case.

4. Computational efficiency

The computational efficiency index of an iterative method for solving a nonlinear system is defined by $CI = \rho^{\frac{1}{C}}$, where ρ is the order of convergence and C is the computational cost of each method. We will examine the computational efficiency of the proposed methods and compare it with that of the methods introduced in [14–16]. To compute the function F and its derivative F' in any iterative method, we evaluate n and n^2 scalar functions, respectively. In addition, we must account for the number of operations shown in Table 2.

As shown in Fig. 1 and Table 3, the method ESS8 and method (41) demonstrate higher computational efficiency than the other methods considered.

5. Numerical Experiments

To validate the theoretical results concerning the convergence behavior and computational efficiency of the proposed methods, we present several numerical experiments and compare their performance with existing methods of the same order. The experiments were made with an Intel Core processor i5-4590, with a CPU of 3.30 GHz and 4096 MB of RAM memory. All computations are performed in the programming package Mathematica 14 using multiple-precision arithmetics. We have used 1000-digit floating-point arithmetic to minimize round-off errors as much as possible. The iterative process is terminated when the following stopping criterion is satisfied:

$$\|x_{k+1} - x_k\| + \|F(x_{k+1})\| \le 10^{-30}.$$

Table 2

Computational cost of different operations

	Computational cost
LU decomposition	$\frac{1}{3}(n^3 - n)$
Solution of two triangular systems	n^2
Matrix-matrix multiplication	n^3
Scalar-vector multiplication	n
Component-wise multiplication (division) of vectors	n

Table 3

Comparison of computational efficiency

№	methods	ρ	C_i	CI
1	(8), (10), (22)	8	$C_1 = \frac{1}{3}n^3 + 4n^2 + \frac{50}{3}n$	$8^{1/C_1}$
2	(8), (10), (29)	8	$C_2 = \frac{1}{3}n^3 + 4n^2 + \frac{41}{3}n$	$8^{1/C_2}$
3	(8), (10), (32)	8	$C_3 = \frac{1}{3}n^3 + 4n^2 + \frac{50}{3}n$	$8^{1/C_3}$
4	(8), (10), (33)	8	$C_4 = \frac{1}{3}n^3 + 4n^2 + \frac{44}{3}n$	$8^{1/C_4}$
5	(8), (10), (34)	8	$C_4 = \frac{1}{3}n^3 + 4n^2 + \frac{44}{3}n$	$8^{1/C_4}$
6	(41)	8	$C_6 = \frac{1}{3}n^3 + 4n^2 + \frac{40}{3}n$	$8^{1/C_6}$
7	NLM8	8	$C_7 = \frac{1}{3}n^3 + 13n^2 + \frac{8}{3}n$	$8^{1/C_7}$
8	ZCO8	8	$C_8 = \frac{1}{3}n^3 + 11n^2 + \frac{14}{3}n$	$8^{1/C_8}$

The results of the numerical experiments are presented in Tables 4–7. These tables include the number of iterations k, the elapsed CPU time, the absolute error between consecutive iterates $\|x_{k+1}-x_k\|$, the absolute residual error $\|f(x_{k+1})\|$ of the corresponding function, and the computational order of convergence ρ_{co}. Here, the computational order of convergence ρ_{co} is calculated using the formula

$$\rho_{co} = \frac{\ln(\|F(x_{k+1})\|/\|F(x_k)\|)}{\ln(\|F(x_k)\|/\|F(x_{k-1})\|)}.$$

For the purpose of comparison, our analysis includes the methods proposed in [14–16]. For convenience, we will use the following abbreviations for the methods throughout the remainder of this paper.

- ESS8: Method (8) with (10) and (22), $a = 0$, $b = d = 0$, $\alpha = 0$.
- EWL8: Method (8) with (10) and (29), $a = 0$, $b = d = 0$.
- EBRW8: Method (8) with (10) and (32), $a = 0$, $b = d = 0$.
- EZO8: Method (8) with (10) and (33), $a = 0$, $b = d = 0$, $w_1 = w_3 = 0$, $w_2 = 1$, $\tilde{\beta} = 4$, $\tilde{\gamma} = 3$.
- ECL8: Method (8) with (10) and (34), $a = d = 0$, $d = 1$, $\delta_1 = \delta_3 = 0$, $\delta_2 = 1$.

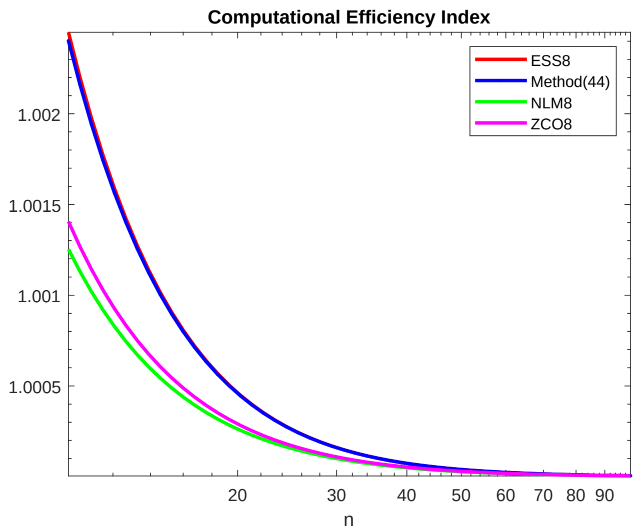

Figure 1. Computational Efficiency Index for n = 10 to 100 (logarithmic scale)

- NOM8: Method (41) with $\beta = \gamma = 0$.
- CTT8: Cordero et al. method (2025).
- NLM8: Sharma et al. method (2017).
- ZCO8: Zhanlav et al. method (2020).

Example 1. We begin with the following system of n equations:

$$\sum_{j=1, j\neq i}^{n} x_j - e^{-x_i} = 0, \quad 1 \leq i \leq n,$$

with $n = 50$. The initial approximation is chosen as $x_0 = (0.5, 0.5, \dots, 0.5)^T$.

Example 2. Our second example is given by the following system of nonlinear equations:

$$\begin{cases} x_i x_{i+1} - e^{-x_i} - e^{-x_{i+1}} = 0, & 1 \leq i \leq n-1, \\ x_n x_1 - e^{-x_n} - e^{-x_1} = 0, \end{cases}$$

where $n = 75$. The initial approximation used in all the methods is $x_0 = (1.2, 1.2, 1.2, 1.2, \dots, 1.2, 1.2)^t$.

Example 3. We consider the system of trigonometric equations:

$$x_i - \cos\left(2x_i - \sum_{j=1}^{n} x_j\right) = 0, \quad i = 1, 2, \dots, n,$$

with $n = 100$. The exact solution is $x^* = \{0.2062, 0.2062 \dots, 0.2062\}^T$. To approximate this solution, we choose the initial vector $x_0 = \{0.3, 0.3, \dots, 0.3\}^T$.

Table 4

Results Example 1

Method	CPU Time	Iterations	$\|x_{k+1} - x_k\|$	$\|F(x_{k+1})\|$	ACOC
ESS8	0.312	3	8.3528×10^{-111}	2.8275×10^{-895}	8.00
EWL8	0.344	3	2.0945×10^{-110}	5.0227×10^{-892}	8.00
EBRW8	0.329	3	1.4007×10^{-110}	3.5560×10^{-895}	8.00
EZO8	0.328	3	8.5898×10^{-111}	1.9078×10^{-893}	8.00
ECL8	0.312	3	9.2848×10^{-107}	1.8216×10^{-862}	8.00
NOM8	0.329	3	2.7848×10^{-108}	7.3470×10^{-875}	8.00
CTT8	0.328	3	3.7717×10^{-108}	1.0796×10^{-873}	8.00
NLM8	1.625	3	1.4001×10^{-123}	6.9822×10^{-992}	8.00
ZCO8	1.812	3	4.7542×10^{-125}	7.1425×10^{-999}	8.00

Table 5

Results Example 2

Method	CPU Time	Iterations	$\|x_{k+1} - x_k\|$	$\|F(x_{k+1})\|$	ACOC
ESS8	0.93	3	7.1752×10^{-81}	1.1021×10^{-654}	8.00
EWL8	0.109	3	1.4674×10^{-82}	1.7636×10^{-668}	8.00
EBRW8	0.94	3	4.6937×10^{-73}	1.7645×10^{-591}	8.00
EZO8	0.95	3	1.4790×10^{-78}	3.1931×10^{-636}	8.00
ECL8	0.94	3	9.3866×10^{-69}	6.8655×10^{-555}	8.00
NOM8	0.78	3	7.8886×10^{-57}	3.4590×10^{-458}	8.00
CTT8	0.94	3	7.7321×10^{-59}	1.5257×10^{-474}	8.00
NLM8	1.375	3	1.4355×10^{-94}	9.4181×10^{-540}	8.00
ZCO8	1.102	3	2.312×10^{-91}	1.7645×10^{-538}	8.00

Example 4. Finally, we analyze the performance of the methods on a large-scale nonlinear system:

$$\begin{cases} x_i^2 x_{i+1} - 1 = 0, & 1 \le i \le n-1, \\ x_n^2 x_1 - 1 = 0, \end{cases}$$

with $n = 1000$. The exact solution is $x^* = (1, 1, \dots, 1)^T$ and $x_0 = (1.25, 1.25, \dots, 1.25)^t$ is the initial vector used.

The results obtained from our experiments provide complete support for the convergence theory presented in Sections 2 and 3. Additionally, the methods listed above were compared in terms of CPU

Table 6

Results Example 3

Method	CPU Time	Iterations	$\|x_{k+1} - x_k\|$	$\|F(x_{k+1})\|$	ACOC
ESS8	6.594	3	1.2652×10^{-47}	3.0069×10^{-370}	8.00
EWL8	6.688	3	7.9068×10^{-48}	1.1668×10^{-375}	8.00
EBRW8	6.672	3	1.8246×10^{-45}	5.5836×10^{-354}	8.00
EZO8	6.657	3	2.3788×10^{-48}	5.4431×10^{-376}	8.00
ECL8	6.594	3	6.3970×10^{-46}	5.1656×10^{-368}	8.00
NOM8	6.687	3	1.1417×10^{-38}	1.6423×10^{-273}	8.00
CTT8	6.634	3	3.9494×10^{-39}	2.8353×10^{-277}	8.00
NLM8	21.092	3	1.0486×10^{-42}	2.4170×10^{-330}	8.00
ZCO8	19.304	3	4.5715×10^{-40}	3.1475×10^{-328}	8.00

Table 7

Results Example 4

Method	CPU Time	Iterations	$\|x_{k+1} - x_k\|$	$\|F(x_{k+1})\|$	ACOC
ESS8	27.969	3	1.1391×10^{-41}	2.9879×10^{-338}	8.00
EWL8	28.453	3	7.4365×10^{-45}	3.9437×10^{-364}	8.00
EBRW8	37.765	3	3.4087×10^{-38}	8.8383×10^{-310}	8.00
EZO8	28.375	3	1.1741×10^{-38}	1.2945×10^{-313}	8.00
ECL8	28.625	3	2.1198×10^{-39}	5.0136×10^{-320}	8.00
NOM8	28.766	3	3.6141×10^{-31}	4.9968×10^{-271}	8.00
CTT8	28.984	3	2.8237×10^{-32}	4.2326×10^{-272}	8.00
NLM8	633.995	3	1.1622×10^{-33}	2.3326×10^{-277}	7.99
ZCO8	629.449	3	7.2358×10^{-32}	1.5687×10^{-272}	7.99

time. As shown in Tables 4–7, the proposed method ESS8 demonstrates superior speed compared to the other methods. It is worth noting that iteration (8) with $a = 2$, $b = d = 0$ yields similar results in the same experiments. Finally, based on our experimental results, we conclude that methods with vector and scalar coefficients require less CPU time than other well-known methods with matrix coefficients.

Conclusions

The main contributions of this paper are as follows:

- Investigation in R^n with point-wise multiplication and division not only allows us to derive simple schemes but also to design extensions of many eighth-order iterations to multidimensional case.
- These extensions maintained the optimality properties of the original methods.
- We propose first wide class of optimal eighth-order iterative methods with vector and scalar coefficients for solving systems of nonlinear equations.
- As a whole, the results obtained in this paper, can be considered as new achievement in iteration theory.

In conclusion, the proposed methods fulfill the fundamental criteria of high-quality algorithms: low computational cost, minimal execution time, and a simple structure.

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Оптимальные трёхшаговые итерационные методы восьмого порядка для решения систем нелинейных уравнений

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Аннотация. В данной статье мы впервые предлагаем расширение оптимальных методов восьмого порядка на многомерный случай. Показано, что эти расширения сохранили свойства оптимальности исходных методов. Вычислительная эффективность предлагаемых методов сравнивается с известными методами. Проводится сравнение с другими методами. Для подтверждения теоретических результатов и эффективности методов включены численные эксперименты.

Ключевые слова: методы ньютоновского типа, системы нелинейных уравнений, порядок сходимости, оптимальность и расширение методов, индекс эффективности