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DECOMPOSITION OF CANONICAL REPRESENTATIONS ON THE LOBACHEVSKY PLANE ASSOCIATED WITH LINEAR BUNDLES

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Abstract. We decompose canonical representations on the Lobachevsky plane, associated with sections of linear bundles
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Introduction

In our work [1] we described canonical and boundary representations of the group $G = \mathrm{SU}(1,1)$ on the Lobachevsky plane D in sections of linear bundles on D. Now we decompose these representations into irreducible ones. We lean on works [2], [3].

1. Representations of $\mathrm{SU}(1,1)$ induced by characters of $\mathrm{U}(1)$

The Lobachevsky plane is the unit disk $D: z\overline{z} < 1$ on the complex plane with the linear-fractional action of G:

$$z \mapsto z \cdot g = \frac{az+\overline{b}}{bz+\overline{a}}, \quad g = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}, \quad a\overline{a} - b\overline{b} = 1.$$

The boundary S of D is the circle $z\overline{z} = 1$, it consists of points $s = \exp i\alpha$, the measure ds on S is $d\alpha$. Let $\overline{D}$ be the closure of D: $\overline{D} = D \cup S$. Let

$$p = 1 - z\overline{z},$$

so that $D = \{p > 0\}$ and $S = \{p = 0\}$. The stabilizer of the point $z = 0$ is the maximal compact subgroup $K = \mathrm{U}(1)$ consisting of diagonal matrices:

$$k = \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix}, \quad a\overline{a} = 1,$$

so that $D = G/K$. The Euclidean measure $dxdy$ on D is $(1/2)\,dp\,ds$, a G-invariant measure $d\mu(z)$ on D is

$$d\mu(z) = p^{-2}dxdy.$$

If M is a manifold, then $\mathcal{D}(M)$ denotes the Schwartz space of compactly supported infinitely differentiable $\mathbb{C}$-valued functions on M, with a usual topology, and $\mathcal{D}'(M)$ denotes the space of distributions on M – of antilinear continuous functionals on $\mathcal{D}(M)$.

Recall principal non-unitary series representations of G trivial on the center. Let $\sigma \in \mathbb{C}$. The representation T_σ acts on the space $\mathcal{D}(S)$ by

$$(T_\sigma(g)\varphi)(s) = \varphi(s \cdot g)|bs + \overline{a}|^{2\sigma}.$$

The inner product from $L^2(S, ds)$:

$$\langle \psi, \varphi \rangle_S = \int_S \psi(u)\overline{\varphi(u)}ds(u) \tag{1.1}$$

is invariant with respect to the pair $(T_\sigma, T_{-\overline{\sigma}-1})$.

If $\sigma \notin \mathbb{Z}$, then T_σ is irreducible and equivalent to $T_{-\sigma-1}$ (for $\sigma \in \mathbb{Z}$ there is a "partial equivalence").

The following operator A_σ acts on $\mathcal{D}(S)$ and intertwines T_σ and $T_{-\sigma-1}$:

$$\left(A_\sigma\varphi\right)(s) = \int_S |1 - s\overline{u}|^{-2\sigma-2}\,\varphi(u)\,du,$$

exponents $\psi_n(s) = s^n$ are eigenfunctions for A_σ with eigenvalues $a_n(\sigma)$:

$$a_n(\sigma) = 2\pi\,(-1)^n\,\frac{\Gamma(-2\sigma-1)}{\Gamma(-\sigma+n)\,\Gamma(-\sigma-n)}\,.$$

The composition $A_\sigma A_{-\sigma-1}$ is a scalar operator:

$$A_\sigma A_{-\sigma-1} = \frac{1}{2\pi\omega(\sigma)} \cdot E$$

where $\omega(\sigma)$ is a "Plancherel measure" (see Theorem 1.1):

$$\omega(\sigma) = \frac{1}{2\pi^2}\Big(\sigma + \frac{1}{2}\Big)\cot\sigma\pi,$$

The operator A_σ is meromorphic in σ with simple poles at $\sigma \in -(1/2) + \mathbb{N}$.

There are four series of unitarizable irreducible representations: the *continuous series*: T_σ, $\sigma = -(1/2) + i\rho$, $\rho \in \mathbb{R}$, an inner product is (1.1); the *complementary series*: T_σ, $-1 < \sigma < 0$, an inner product is the form $\langle A_\sigma\psi, \varphi\rangle_S$ with a suitable factor; the *holomorphic* and *antiholomorphic* series consisting of subfactors $T_{\sigma,\pm}$ of T_σ, $\sigma \in \mathbb{Z}$.

We shall use denotation:

$$z^{\mu,\,m} = |z|^\mu \left(\frac{z}{|z|}\right)^m, \quad \mu \in \mathbb{C}, \quad m \in \mathbb{Z}.$$

Let us take characters (one dimensional representations) of the group K that are trivial on the center $\pm E$, namely,

$$\omega_m(k) = \overline{a}^{2m} = a^{-2m}, \quad k \in K, \quad m \in \mathbb{Z}.$$

Denote by $U^{(m)}$ the representation of the group G induced by the character ω_m. It acts by translations on the space $\mathcal{D}^{(m)}(G)$ of functions $\psi \in \mathcal{D}(G)$ satisfying the condition $\psi(kg) = \omega_m(k)\,\psi(g)$. It can be realized on functions on the disk D:

$$\left(U^{(m)}(g)f\right)(z) = f(z\cdot g)\,(bz+\overline{a})^{0,2m}.$$

The representation $U^{(m)}$ moves the Casimir element of the Lie algebra $\mathfrak{g}$ to the Casimir operator (a differential operator on D). Its radial part is the following differential operator on $[1,\,\infty)$:

$$L_m = (c^2-1)\,\frac{d^2}{dc^2} + 2c\,\frac{d}{dc} + \frac{2m^2}{c+1}. \tag{1.2}$$

The representation $U^{(m)}$ preserves the inner product

$$(f,\,h)_{d\mu} = \int_D f(z)\,\overline{h(z)}\,d\mu(z).$$

We denote the unitary completion of $U^{(m)}$ acting on $L^2(D, d\mu)$ by the same symbol.

Let $\mathcal{D}(\overline{D})$ be the space of restrictions to $\overline{D}$ of functions from $\mathcal{D}(\mathbb{C})$ with the induced topology, and by $\mathcal{D}'(\overline{D})$ the space of distributions on $\mathbb{C}$ with supports in $\overline{D}$. Consider the inner product with respect to the Lebesgue measure on D:

$$\langle F, f\rangle_D = \int_D F(z)\overline{f(z)}dxdy, \quad z = x+iy. \tag{1.3}$$

The space $\mathcal{D}(\overline{D})$ can be embedded into $\mathcal{D}'(\overline{D})$ by assigning to $h \in \mathcal{D}(\overline{D})$ the functional $f \mapsto \langle h, f\rangle_D$, $f \in \mathcal{D}(\overline{D})$. So we shall write the value of $F \in \mathcal{D}'(\overline{D})$ at $f \in \mathcal{D}(\overline{D})$ in the same form: $\langle F, f\rangle_S$.

We define the Poisson transform $P_\sigma^{(m)} : \mathcal{D}(S) \to C^\infty(D)$ and the Fourier transform $F_\sigma^{(m)} : \mathcal{D}(D) \to \mathcal{D}(S)$, associated to the character ω_m, as integral operators

$$\left(P_\sigma^{(m)}\varphi\right)(z) = p^{-\sigma}\int_S (1-s\overline{z})^{2\sigma,-2m}\, s^m\, \varphi(s)\,ds.$$

$$\left(F_\sigma^{(m)}f\right)(s) = s^{-m}\int_D (1-s\overline{z})^{2\sigma,2m}\, p^{-\sigma}\, f(z)d\mu(z).$$

The Poisson and Fourier transforms $P_\sigma^{(m)}$ and $F_\sigma^{(m)}$ intertwine representations $T_{-\sigma-1}$ with $U^{(m)}$ and $U^{(m)}$ with T_σ respectively. The Poisson and the Fourier transform are conjugate to each other:

$$\langle F_\sigma^{(m)} f,\, \varphi\rangle_S = (f,\, P_{\overline{\sigma}}^{(m)}\varphi)_{d\mu}.$$

Using the spectral resolution of the radial part of the Casimir operator (1.2), we obtain the following Plancherel theorem for $U^{(m)}$.

Theorem 1.1. *Let us assign to a function $f \in \mathcal{D}(D)$ the family $\{F_\sigma^{(m)} f\}$ where $\sigma = -1/2 + i\rho$, $\rho \in \mathbb{R}$, of its Fourier components of the continuous series and the family $\{F_k^{(m)}, f\}$ where $k = 0, 1, \ldots, |m|-1$, of its Fourier components of the analytic (if $m < 0$) or the anti-analytic (if $m > 0$) series. This correspondence is G-equivariant. One has the inversion formula:*

$$\begin{aligned} f(z) &= \int_{-\infty}^{\infty} \omega(\sigma)\, (P_{-\sigma-1}^{(m)}\, F_\sigma^{(m)}\, f)(z)\big|_{\sigma=-1/2+i\rho}\, d\rho \\ &+ \sum_{k=0}^{|m|-1} \frac{1}{2\pi^2}\, (2k+1)\, (P_{-k-1}^{(m)}\, F_k^{(m)}\, f)(z), \end{aligned}$$

and the Plancherel formula for functions $f, h \in \mathcal{D}(D)$:

$$\begin{aligned} (f,\, h)_{d\mu} &= \int_{-\infty}^{\infty} \omega(\sigma)\, \langle F_\sigma^{(m)} f,\, F_\sigma^{(m)} h\rangle_S\big|_{\sigma=-1/2+i\rho}\, d\rho \\ &+ \sum_{k=0}^{|m|-1} \frac{1}{2\pi^2}\, \langle F_k^{(m)} f,\, F_{-k-1}^{(m)} h\rangle_S. \end{aligned} \tag{1.4}$$

Therefore, the previous correspondence can be extended from the space $\mathcal{D}(S)$ to $L^2(D, d\mu)$ and gives then the decomposition of the unitary representation $U^{(m)}$ on $L^2(D, d\mu)$ into the direct integral of the representations T_σ, $\sigma = -1/2 + i\rho$ of the continuous series, and the direct sum of $|m|$ representations $T_{k,+}$ or $T_{k,-}$, $k = 0, 1, \ldots, |m|-1$, of the analytic ($m > 0$) or anti-analytic ($m > 0$) series. This decomposition is multiplicity free.

2. Canonical representations

Let $\lambda \in \mathbb{C}$. We define the *canonical representation* $R_{\lambda,m}$ of the group G associated with a character of K as follows:

$$(R_{\lambda,m}(g)f)\,(z) = f(z \cdot g)\,(bz + \overline{a})^{-2\lambda-4,2m},$$

it acts on the space $\mathcal{D}(\overline{D})$.

The inner product (1.3) is invariant with respect to the pair $(R_{\lambda,m},\, R_{-\overline{\lambda}-2,m})$:

$$\langle R_{\lambda,m}(g)f,\, h\,\rangle_D = \langle f,\, R_{-\overline{\lambda}-2,m}(g^{-1})h\,\rangle_D, \quad g \in G. \tag{2.1}$$

Let us define the operator $Q_{\lambda,m}$ – first on $\mathcal{D}(D)$:

$$(Q_{\lambda,m} f)\,(z) = c(\lambda, m) \int_D (1 - z\overline{w})^{2\lambda,2m}\, f(w)\,du\,dv,$$

where

$$c(\lambda, m) = \frac{-\lambda + m - 1}{\pi}.$$

It intertwines $R_{\lambda,m}$ and $R_{-\lambda-2,m}$:

$$Q_{\lambda,m}\, R_{\lambda,m}(g) = R_{-\lambda-2,m}(g)\, Q_{\lambda,m}, \quad g \in G,$$

and interacts with the form (1.3) as follows:

$$\langle Q_{\lambda,m} f,\, h\,\rangle_D = \langle f,\, Q_{\overline{\lambda},m}\, h\,\rangle_D. \tag{2.2}$$

The formulae (2.1) and (2.2) allow to extend the representation $R_{\lambda,m}$ and the operator $Q_{\lambda,m}$ to the space $\mathcal{D}'(\overline{D})$ of distributions on $\overline{D}$.

Canonical representations $R_{\lambda,m}$ generate boundary representations $L_{\lambda,m}$ and $M_{\lambda,m}$. Consider the Taylor series of $f \in \mathcal{D}(\overline{D})$ in powers of p:

$$f(z) \sim a_0 + a_1\, p + a_2\, p^2 + \cdots,$$

where $a_k = a_k(s)$ are functions in $\mathcal{D}(S)$:

$$a_k(s) = \frac{1}{k!}\left(\frac{\partial}{\partial p}\right)^k\Big|_{p=0} f(z).$$

Let $a(f)$ denote the column $(a_0, a_1, \ldots)$ of the Taylor coefficients.

Denote by $\Sigma_k(\overline{D})$ the space of distributions on $\mathbb{C}$ concentrated at S and of the form

$$\zeta = \varphi_0(s)\,\delta(p) + \varphi_1(s)\,\delta'(p) + \cdots + \varphi_k(s)\,\delta^{(k)}(p),$$

where $\delta(p)$ is the Dirac delta function on the real line (being a continuous linear functional on $\mathcal{D}(\mathbb{R})$) and $\delta^{(j)}(p)$ its j-th derivative. Set

$$\Sigma(\overline{D}) = \cup_{k=0}^{\infty} \Sigma_k(\overline{D}).$$

There is a natural filtration

$$\Sigma_0(\overline{D}) \subset \Sigma_1(\overline{D}) \subset \Sigma_2(\overline{D}) \subset \cdots \tag{2.3}$$

A distribution $\varphi(s)\,\delta^{(l)}(p)$ acts on a function $f \in \mathcal{D}(\overline{D})$ as follows:

$$\langle \varphi(s)\,\delta^{(l)}(p),\, f\rangle_D = \frac{1}{2}(-1)^l\, l!\,\langle \varphi,\, a_l\rangle_S. \tag{2.4}$$

Distributions from $\Sigma_k(\overline{D})$ can be extended to a wider space than $\mathcal{D}(\overline{D})$. Namely, let $\mathcal{T}_k(\overline{D})$ be the space of functions f on $\overline{D}$ of class C^∞ on D and on S and having a Taylor decomposition of order k:

$$f(z) = a_0 + a_1 p + a_2 p^2 + \ldots + a_k p^k + o(p^k)$$

uniformly with respect to $u \in S$, where $a_m = a_m(f)$ belong to $\mathcal{D}(S)$. Then (2.4) is well preserved for $f \in \mathcal{T}_k(\overline{D})$.

The canonical representation $R_{\lambda,m}$ acting on $\mathcal{D}'(\overline{D})$, preserves the space $\Sigma\,(\overline{D})$ and the filtration (2.3). The first boundary representation $L_{\lambda,m}$ is the restriction of $R_{\lambda,m}$ to $\Sigma\,(\overline{D})$. The second boundary representation $M_{\lambda,m}$ acts on columns $a(f)$ by:

$$M_{\lambda,m}(g)\,a(f) = a(R_{\lambda,m}(g)f).$$

Theorem 2.1. *The representation $L_{\lambda,m}$ is equivalent to a upper triangular matrix with diagonal $T_{-\lambda-1}, T_{-\lambda}, T_{-\lambda+1}, \ldots$. The equivalence is given by multiplication of the functions $\varphi_k(s)$ by s^{-m}. The representation $M_{\lambda,m}$ is equivalent to a lower triangular matrix with diagonal $T_{-\lambda-2}, T_{-\lambda-3}, \ldots$. The equivalence is given by multiplication of the Taylor coefficients $a_k(s)$ by s^{-m}.*

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. In the generic case: $2\lambda \notin \mathbb{N}$, the representation $L_{\lambda,m}$ is diagonalizable, which means that the space $\Sigma(\overline{D})$ is the direct sum of the spaces $V_{\lambda,k}^{(m)}$ $(k \in \mathbb{N})$, so that $L_{\lambda,m}$ is the direct sum of the $T_{-\lambda-1+k}$ $(k \in \mathbb{N})$.

3. Poisson transform

Let λ, $\sigma \in \mathbb{C}$ and $m \in \mathbb{Z}$. We define the *Poisson transform associated with the canonical representation* $R_{\lambda,m}$ as the map $P_{\lambda,\sigma}^{(m)} : \mathcal{D}(S) \to C^\infty(D)$ by the following formula

$$\left(P_{\lambda,\sigma}^{(m)}\,\varphi\right)(z) = p^{-\lambda-\sigma-2} \int_S (1 - s\overline{z})^{2\sigma,-2m}\, s^m\, \varphi(s)\, ds.$$

The Poisson transform $P_{\lambda,\sigma}^{(m)}$ intertwines the representations $T_{-\sigma-1}$ and the canonical representation $R_{\lambda,m}$:

$$R_{\lambda,m}(g)\, P_{\lambda,\sigma}^{(m)} = P_{\lambda,\sigma}^{(m)}\, T_{-\sigma-1}(g) \quad (g \in G).$$

With the intertwining operators A_σ and $Q_{\lambda,m}$ the Poisson transform interacts as follows:

$$\begin{aligned} P_{\lambda,\sigma}^{(m)}\, A_\sigma &= a_{-m}(\sigma)\, P_{\lambda,-\sigma-1}^{(m)}, \\ Q_{\lambda,m}\, P_{\lambda,\sigma}^{(m)} &= \Lambda^{(m)}(\lambda,\sigma)\, P_{-\lambda-2,\sigma}^{(m)}, \end{aligned}$$

where

$$\Lambda^{(m)}(\lambda,\sigma) = \frac{\Gamma(-\lambda+\sigma)\,\Gamma(-\lambda-\sigma-1)}{\Gamma(-\lambda-m)\,\Gamma(-\lambda+m-1)}\,.$$

The Poisson transform $P_{\lambda,\sigma}^{(m)}$ is meromorphic in σ, and has poles at the points

$$\sigma = \lambda - k, \quad \sigma = -\lambda - 1 + l \quad (k, l \in \mathbb{N}). \tag{3.1}$$

All poles are simple except in the case when the two sequences (3.1) have a non-empty intersection and the pole belongs to this intersection. This happens when $2\lambda + 1 \in \mathbb{N}$ and

$0 \leqslant k, l \leqslant 2\lambda+1$, $k+l = 2\lambda+1$. In this case the pole μ is of the second order. Let us write down the principal part of the Laurent series of $P^{(m)}_{\lambda,\sigma}$ at the poles μ of the first order:

$$P^{(m)}_{\lambda,\sigma} = \frac{\widehat{P}^{(m)}_{\lambda,\mu}}{\sigma - \mu} + \cdots .$$

The residue intertwines $T_{-\mu-1}$ with $R_{\lambda,m}$. Let us write it explicitly. We set

$$V_{\sigma,m,n}(p) = (1-p)^{(m+n)/2}\, F(\sigma+1+m, \sigma+1+n; 2\sigma+2; p),$$

where F is the Gauss hypergeometric function. Expand V in powers of p:

$$V_{\sigma,m,n}(p) = \sum_{k=0}^{\infty} w^{(m)}_{\sigma,k}(n)\, p^k,$$

here $w^{(m)}_{\sigma,k}$ are polynomials in n of degree k. The coefficients of these polynomials are rational functions of σ with simple poles. Now we set

$$W^{(m)}_{\sigma,k} = w^{(m)}_{\sigma,k}\left(\frac{1}{i}\,\frac{d}{d\alpha}\right).$$

If a pole μ belongs only to one of the sequences (3.1), then it is simple and

$$\begin{aligned} \widehat{P}^{(m)}_{\lambda,\lambda-k} &= (-1)^{k+m}\,\frac{1}{k!}\, a_{-m}(\lambda-k)\, \xi^{(m)}_{\lambda,k}, \\ \widehat{P}^{(m)}_{\lambda,-\lambda-1+l} &= (-1)^{l+m}\,\frac{1}{l!}\, \xi^{(m)}_{\lambda,l} \circ A_{\lambda-l}, \end{aligned}$$

where $\xi^{(m)}_{\lambda,k}$ is the following operator $\mathcal{D}(S) \to \Sigma_k(\overline{D})$:

$$\xi^{(m)}_{\lambda,k}\,\varphi = s^m \sum_{n=0}^{k} (-1)^n\, \frac{k!}{(k-n)!}\, \left(W^{(m)}_{\lambda-k,n}\,\varphi\right)(s)\, \delta^{(k-n)}(p). \tag{3.2}$$

The operator $\xi^{(m)}_{\lambda,k}$ is meromorphic in λ. For fixed $k = 1, 2\ldots$ it has k poles (simple) at the points $\lambda = k-1$, $k-3/2$, $k-2, \ldots, (k-1)/2$. It intertwines $T_{-\lambda-1+k}$ with $L_{\lambda,m}$ (restricted to $\Sigma_k(\overline{D})$).

Theorem 3.1. *Up to a factor, the composition of the operators* $Q_{\lambda,m}$ *and* $\xi^{(m)}_{\lambda,k}$ *is the Poisson transform* $P^{(m)}_{-\lambda-2,\lambda-k}$*:*

$$Q_{\lambda,m}\,\xi^{(m)}_{\lambda,k} = q^{(m)}_{\lambda,k} \cdot P^{(m)}_{-\lambda-2,\lambda-k},$$

where

$$\begin{aligned} q^{(m)}_{\lambda,k} &= \frac{1}{2}\,(-1)^{k+m}\, k!\, a_{-m}(-\lambda-1+k)\, \Lambda^{(m)}_k(\lambda), \\ \Lambda^{(m)}_k(\lambda) &= -\,\frac{1}{2\pi^2}\,(2\lambda-2k+1)\,\frac{\Gamma(\lambda+m+1)\,\Gamma(\lambda-m+2)}{k!\,\Gamma(2\lambda+2-k)}. \end{aligned}$$

4. Fourier transform

Let $\lambda,\ \sigma \in \mathbb{C}$ and $m \in \mathbb{Z}$. We define the *Fourier transform associated with the canonical representation* $R_{\lambda,m}$ as the map $F^{(m)}_{\lambda,\sigma} : \mathcal{D}(\overline{D}) \to \mathcal{D}(S)$ by the following formula

$$\left(F^{(m)}_{\lambda,\sigma} f\right)(s) = s^{-m} \int_D (1 - z\overline{s})^{2\sigma,2m}\, p^{\lambda-\sigma}\, f(z)dxdy.$$

The integral converges absolutely for $\mathrm{Re}\,(\lambda-\sigma) > -1$, $\mathrm{Re}\,(\lambda+\sigma) > -2$ and can be meromorphically continued in σ and λ. The Poisson and the Fourier transform are conjugate to each other:

$$\langle F^{(m)}_{\lambda,\sigma} f,\ \varphi\rangle_S = \langle f,\ P^{(m)}_{-\overline{\lambda}-2,\overline{\sigma}}\, \varphi\rangle_D. \tag{4.1}$$

This allows to transfer statements about the Poisson transform to the Fourier transform. The Fourier transform interacts with the intertwining operators as follows:

$$\begin{aligned} A_\sigma\, F^{(m)}_{\lambda,\sigma} &= a_{-m}(\sigma)\, F^{(m)}_{\lambda,-\sigma-1}, \\ F^{(m)}_{-\lambda-2,\sigma}\, Q_{\lambda,m} &= \Lambda^{(m)}(\lambda,\sigma)\, F^{(m)}_{\lambda,\sigma}. \end{aligned}$$

It has poles in σ at the points

$$\sigma = -\lambda - 2 - k, \quad \sigma = \lambda + 1 + l \quad (k, l \in \mathbb{N}). \tag{4.2}$$

All poles are simple, except the case $-2\lambda-3 \in \mathbb{N}$ and the pole μ belongs to both sequences (4.2), i. e. $0 \leqslant k, l \leqslant -2\lambda - 3$ and $k + l = -2\lambda - 3$. In this case μ is of the second order. For the Laurent coefficients of the Fourier transform we use a similar notation as in case of the Poisson transform. The first Laurent coefficient $\widehat{F}^{(m)}_{\lambda,\mu}$ for the first order μ intertwines $R_{\lambda,m}$ with T_μ. Let us write it explicitly:

$$\begin{aligned} \widehat{F}^{(m)}_{\lambda,-\lambda-2-k} &= \frac{1}{2}\,(-1)^m\, a_{-m}(-\lambda-2-k)\, b^{(m)}_{\lambda,k}, \\ \widehat{F}^{(m)}_{\lambda,\lambda+1+l} &= -\frac{1}{2}\,(-1)^m\, A_{-\lambda-2-l}\, b^{(m)}_{\lambda,l}, \end{aligned}$$

where $b^{(m)}_{\lambda,k}$ is a “boundary” operator $\mathcal{D}(\overline{D}) \to \mathcal{D}(S)$ which is defined in terms of the Taylor coefficients c_n of f as follows:

$$b^{(m)}_{\lambda,k}(f) = \sum_{n=0}^{k} W^{(m)}_{-\lambda-2-k,k-n}\,(s^{-m} c_n).$$

The operators $\xi^{(m)}$ and $b^{(m)}$ are conjugate to each other (up to a factor):

$$\langle f,\ \xi^{(m)}_{-\overline{\lambda}-2,k}\, \varphi\rangle_D = \frac{1}{2}\,(-1)^k\, k!\, \langle b^{(m)}_{\lambda,k}(f),\ \varphi\rangle_S.$$

The operator $b^{(m)}_{\lambda,k}$ intertwines $R_{\lambda,m}$ with $T_{-\lambda-2-k}$. It is meromorphic in λ. It has k poles (simple) at the points $\lambda = -k-1, -k-1/2, \ldots, (-k-3)/2$.

5. Decomposition of canonical representations

For simplicity we restrict ourselves to generic λ lying in the strips I_k, $(k \in \mathbb{Z})$.:

$$-3/2+k < \operatorname{Re}\lambda < -1/2+k.$$

Case A: $\lambda \in I_0$. Let $f, h \in \mathcal{D}(\overline{D})$. Consider the functions

$$f_0(z) = p^{\lambda+2} f(z), \quad h_0(z) = p^{-\overline{\lambda}} h(z).$$

Since $\lambda \in I_0$, both functions $f_0(z)$ and $h_0(z)$ belong to $L^2(D, d\mu)$. Let us apply to this pair of functions f_0, h_0 the Plancherel formula (1.19). We obtain:

$$\begin{aligned}(f_0,\, h_0)_{d\mu} &= \int_{-\infty}^{\infty} \omega(\sigma)\, \langle F_\sigma^{(m)} f_0,\, F_{-\overline{\sigma}-1}^{(m)} h_0\rangle_S \Big|_{\sigma=-1/2+i\rho} d\rho \\ &+ \sum_{n=0}^{|m|-1} \frac{1}{2\pi^2}\,(2n+1)\, \langle F_n^{(m)} f_0,\, F_{-n-1}^{(m)} h_0\rangle_S.\end{aligned}$$

Then we return to f and h:

$$\begin{aligned}(f,\, h)_D &= \int_{-\infty}^{\infty} \omega(\sigma)\, \langle F_{\lambda,\sigma}^{(m)} f,\, F_{-\overline{\lambda}-2,-\overline{\sigma}-1}^{(m)} h\rangle_S \Big|_{\sigma=-1/2+i\rho} d\rho \\ &+ \sum_{n=0}^{|m|-1} \frac{1}{2\pi^2}\,(2n+1)\, \langle F_{\lambda,n}^{(m)} f,\, F_{-\overline{\lambda}-2,-n-1}^{(m)} h\rangle_S. \end{aligned} \tag{5.1}$$

Now usimg the conjugacy (4.1), we transfer the Fourier transform of h to the Poisson transform of $F_{\lambda,\sigma}^{(m)} f$. We obtain a formula that gives an expansion of f regarded as a distribution in $\mathcal{D}'(\overline{D})$:

$$\begin{aligned}f &= \int_{-\infty}^{\infty} \omega(\sigma)\, P_{\lambda,-\sigma-1}^{(m)}\, F_{\lambda,\sigma}^{(m)} f \Big|_{\sigma=-1/2+i\rho} d\rho \\ &+ \sum_{n=0}^{|m|-1} \frac{1}{2\pi^2}\,(2n+1)\, P_{\lambda,-n-1}^{(m)}\, F_{\lambda,n}^{(m)} f. \end{aligned} \tag{5.2}$$

Theorem 5.1. *Let $\lambda \in I_0$. Then the canonical representation $R_{\lambda,m}$ decomposes, in a similar way as $U^{(m)}$, see § 1, into the direct integral of the representations T_σ, $\sigma = -1/2+i\rho$, of the continuous series and the direct sum of $|m|$ representations $T_{n,+}$ or $T_{n,-}$, $n = 0, 1, \ldots, |m|-1$, of the analytic ($m<0$) or the anti-analytic series ($m>0$) with multiplicity one. Namely, if we assign to $f \in \mathcal{D}(\overline{D})$ the family of Fourier components $\{F_{\lambda,\sigma}^{(m)} f\}$ where $\sigma = -1/2+i\rho$ and $\sigma \in \{0, 1, \ldots, |m|-1\}$, then this correspondence is G-equivariant. There is an inversion formula* (5.2) *and a decomposition* (5.1) *of the form* $(f,\, h)_D$.

Case B: $\lambda \in I_{k+1},\ k \in \mathbb{N}$. We perform analytic continuation of (5.2) from the strip I_0 to the right, to the strip I_{k+1}. Here the poles of the Poisson transform intersect the line of integration $\operatorname{Re}\sigma = -1/2$ and give additional terms. We obtain

$$f = \int_{-\infty}^{\infty} + \sum_{n=0}^{|m|-1} + \sum_{v=0}^{k} \pi_{\lambda,v}^{(m)}(f), \tag{5.3}$$

where the integral and the first sum mean the same as in (5.2) and

$$\pi_{\lambda,v}^{(m)} = 2\,(-1)^{v+m}\,\frac{1}{v!}\,\frac{1}{a_{-m}(-\lambda-1+v)}\,\xi_{\lambda,v}^{(m)} \circ F_{\lambda,-\lambda-1+v}^{(m)}.$$

The operators $\pi_{\lambda,v}^{(m)},\ v \leqslant k,$ can be extended to $\Sigma_k(\overline{D})$, because the Fourier transforms occuring in $\pi_{\lambda,v}^{(m)}$ are already extended. Thus, the operators $\pi_{\lambda,v}^{(m)},\ v \leqslant k,$ are defined on the space

$$\mathcal{D}_k(\overline{D}) = \mathcal{D}(\overline{D}) + \Sigma_k(\overline{D}). \tag{5.4}$$

The operators $\pi_{\lambda,v}^{(m)},\ v \leqslant k,$ acting on the space $\mathcal{D}_k(\overline{D})$, are projection operators onto the spaces $V_{\lambda,v}^{(m)}$, see § 2 for them, i.e. the following relations hold:

$$\begin{aligned} \pi_{\lambda,v}^{(m)}\,\pi_{\lambda,v}^{(m)} &= \pi_{\lambda,v}^{(m)}, \\ \pi_{\lambda,v}^{(m)}\,\pi_{\lambda,s}^{(m)} &= 0, \quad v \neq s. \end{aligned}$$

Thus, in Case B we have

Theorem 5.2. *Let $\lambda \in I_{k+1},\ k \in \mathbb{N}$. Then the space $\mathcal{D}(\overline{D})$ has to be completed to the space $\mathcal{D}_k(\overline{D})$, see (5.4). On this space the canonical representation $R_{\lambda,m}$ splits into the sum of two terms: the first term decomposes as $R_{\lambda,m}$ does in Case A, the second term decomposes into the sum of the irreducible representations $T_{-\lambda-1+v} \sim T_{\lambda-v}$ with $v = 0,1,\ldots,k$. Namely, let us assign to any $f \in \mathcal{D}_k(\overline{D})$ the family $\{F_{\lambda,\sigma}^{(m)}\}$ where $\sigma = -1/2 + i\rho$, $\sigma = n$, $n = 0,1,\ldots,|m|-1$, and $\sigma = -\lambda-1+v$, $v = 0,1,\ldots,k$. This correspondence is G-equivariant. The function f is recovered by the inversion formula (5.3).*

Case C: $\lambda \in I_{-k-1},\ k \in \mathbb{N}$. Now we perform analytic continuation of (5.2) to the left, to the strip I_{-k-1}. Here the poles

$$\sigma = -\lambda - 2 - v,\ \sigma = \lambda + 1 + v,\ v \in \mathbb{N},\ v \leqslant k,$$

of the integrand (they are poles of the Fourier transform) intersect the line of integration $\operatorname{Re}\sigma = -1/2$ and give additional terms. We obtain

$$f = \int_{-\infty}^{\infty} + \sum_{n=0}^{|m|-1} + \sum_{v=0}^{k} \Pi_{\lambda,v}^{(m)}(f), \tag{5.5}$$

where the integral and the first sum have the same meaming as in (5.2) and

$$\Pi_{\lambda,v}^{(m)} = (-1)^m \frac{1}{a_{-m}(\lambda+1+v)} P_{\lambda,\lambda+1+v}^{(m)} \circ \xi_{\lambda,v}^{(m)}.$$

Denote by $\mathcal{P}_{\lambda,v}^{(m)}$ the image of the operator $P_{\lambda,\lambda+1+v}^{(m)}$. The operators $\Pi_{\lambda,v}^{(m)}$ with $v \leqslant k$ can be extended to the space $\mathcal{T}_k(\overline{D})$ since the operators $b_{\lambda,v}^{(m)}$ with $v \leqslant k$ are defined on this space. In particular, $\Pi_{\lambda,v}^{(m)}$ can be applied to $\mathcal{P}_{\lambda,s}^{(m)}$, $s \leqslant k$, and we can consider the products $\Pi_{\lambda,v}^{(m)} \Pi_{\lambda,s}^{(m)}$ with $v, s \leqslant k$.

Theorem 5.3. *The operators* $\Pi_{\lambda,v}^{(m)}$, $v \leqslant k$, *are projection operators on* $\mathcal{P}_{\lambda,v}^{(m)}$, *namely, the following relations hold:*

$$\begin{aligned} \Pi_{\lambda,v}^{(m)} \Pi_{\lambda,v}^{(m)} &= \Pi_{\lambda,v}^{(m)}, \\ \Pi_{\lambda,v}^{(m)} \Pi_{\lambda,s}^{(m)} &= 0, \quad s \neq v. \end{aligned}$$

Thus, in Case C we have

Theorem 5.4. *Let* $\lambda \in I_{-k-1}$, $k \in \mathbb{N}$. *Then the canonical representation* $R_{\lambda,m}$ *considered on the space* $\mathcal{T}_k(\overline{D})$ *splits into the sum of two terms. The first term acts on the subspace of functions* f *such that their Taylor coefficients* $c_v(f)$ *are equal to zero for* $v \leqslant k$, *and decomposes as* $R_{\lambda,m}$ *in Case A, the second term decomposes into the direct sum of the* $k+1$ *irreducible representations* $T_{-\lambda-2-v}$ $(\sim T_{\lambda+1+v})$, $v = 0, 1, \ldots, k$, *acting on the sum of the spaces* $\mathcal{P}_{\lambda,v}^{(m)}$. *One has an inversion formula, see* (5.5).

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РАЗЛОЖЕНИЕ КАНОНИЧЕСКИХ ПРЕДСТАВЛЕНИЙ НА ПЛОСКОСТИ ЛОБАЧЕВСКОГО В СЕЧЕНИЯХ ЛИНЕЙНЫХ РАССЛОЕНИЙ

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Аннотация. Мы разлагаем канонические представления, действующие в сечениях линейных расслоений на плоскости Лобачевского

Ключевые слова: плоскость Лобачевского; канонические представления; обобщенные функции; граничные представления; преобразования Пуассона и Фурье

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