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Proof of Brouwer's Conjecture (BC) for all graphs with number of vertices $n > n_0$ assuming that BC holds for $n \leq n_0$ for some $n_0 \leq 10^{24}$

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Abstract. In the article, the authors consider the problem of constructing an upper bound for the sum of the maximal eigenvalues of Laplacian of a graph. The article is devoted to proving the Brouwer conjecture, which states that the sum of the t-maximal eigenvalues of Laplacian of a graph does not exceed the number of edges of the graph plus $(t+1)t/2$. Note that we prove the validity of the general Brouwer conjecture under the assumption that the conjecture is valid for a finite number of graphs with the number of vertices less than 10^{24}, i.e., a complete proof of the conjecture is reduced to establishing its validity for a finite number of graphs. The proof of this conjecture attracts the interest of a large number of specialists. There are a number of results for special graphs and a proof of the conjecture for almost all random graphs. The proof we are considering uses an inductive method that has some peculiarities. The original method involves constructing various estimates for the eigenvalues of Laplacian of a graph which is used to construct the induction step. Several variants of the method are considered depending on the values of the coordinates of the eigenvectors of the Laplacian. The well-known fact of equivalence of the validity of the Brouwer conjecture for the graph itself and the complement of the graph is used.

Keywords: Laplacian of graph, eigenvalues

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НАУЧНАЯ СТАТЬЯ

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Доказательство гипотезы Брауэра (ГБ) для всех графов с числом вершин $n > n_0$ в предположении, что ГБ выполняется при $n \leq n_0$ для некоторого $n_0 \leq 10^{24}$

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Аннотация. В работе рассматривается проблема построения верхней оценки для суммы максимальных собственных чисел лапласиана графа. Статья посвящена доказательству гипотезы Брауэра, которая состоит в том, что сумма t-максимальных собственных чисел лапласиана графа не превышает числа ребер графа плюс $(t+1)t/2$. Отметим, что мы доказываем справедливость общей гипотезы Брауэра в предположении справедливости гипотезы для конечного числа графов с числом вершин меньше 10^{24}, т.е. полное доказательство гипотезы сводится к установлению ее справедливости для конечного числа графов. Доказательство данной гипотезы привлекает интерес большого числа специалистов. Имеется ряд результатов для специальных графов и доказательство справедливости гипотезы для почти всех случайных графов. Рассматриваемое нами доказательство использует индуктивный метод, имеющий ряд особенностей. Оригинальный метод предполагает построение различных оценок для собственных чисел лапласиана, который используется для построения шага индукции. Рассматриваются несколько вариантов метода в зависимости от величин координат собственных векторов лапласиана. Используется известный факт эквивалентности справедливости гипотезы Брауэра для самого графа и дополнения графа.

Ключевые слова: лапласиан графа, собственные значения

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Introduction

Let A be $n \times n$ incidence matrix of simple undirected graph G:

$$a_{i,j} = \begin{cases} 1, & \text{iff } (i,j) \in G, \\ 0, & \text{otherwise.} \end{cases}$$

Define the Laplacian $L(G)$ of G as follows

$$L(G) = D - A,$$

where diagonal $n \times n$ matrix D has entries

$$d_i = |\{j : \ (i,j) \in G\}|.$$

We have $\sum_i d_i = 2m$, were m is number of edges in G. Considering G as directed graph with some choice of ordering of vertices in G define $m \times n$ matrix B:

$$b_{i,j} = \begin{cases} 1, & \text{if } j \text{ heard vertex in edge } i, \\ -1, & \text{if } j \text{ tail vertex in edge } i, \\ 0, & \text{otherwise.} \end{cases}$$

Then $L(G) = B^T B$ and hence eigenvalues of matrix $L(G)$ are nonnegative:

$$0 = \mu_n(L(G)) \leq \mu_{n-1}(L(G)) \leq \ldots \leq \mu_1(L(G)).$$

Brouwer's Conjecture 1 (see [1]). *For every graph $G \subset \binom{[n]}{2}$ and integer $t \in [n-1]$, the following inequality is valid:*

$$S_t(G) = \sum_{i=1}^{t} \mu_i(L(G)) \leq m + \binom{t+1}{2}, \ t \in [n].$$

In this article we prove the validness of this conjecture under the assumption than it is true for all $n \leq n_0$ where $n_0 \leq 10^{24}$.

For convenience, we denote

$$\Delta_t(G) = S_t(G) - m(G) - \binom{t+1}{2}.$$

Whenever $\Delta_t(G) \leq 0$, we say that G satisfy BC_t.

It is known to be valid for trees [2], for $k = 1, 2, n-1, n$, for unicyclic and bicyclic graphs [3], for regular graphs [4], for $n \leq 10$ it was checked by A. Brouwer using a computer. In [5] was proved that Brouwer's conjecture holds asymptotically almost surely.

Before the proof of Brouwer‘s conjecture under above assumption (we call it below “Conjecture”) we introduce some consequences of its validity.

Define set of conjugate degrees

$$d^*(G) = \{d_1^*, \ldots, d_n^*\}, \ d_i^* = |\{j : d_j \geq i\}|.$$

We say that the set E of edges is compressed if from $e = (i < j) \in E$ it follows that $e = (i_1 < j_1) \in E$, were $i_1 \leq i$, $j_1 \leq j$. V. Chátal and P. Hammer in [6] introduce the notion of threshold graph. It can be defined as a graph isomorphic (up to permutations on vertices $[n]$) to a graph with compressed edge set.

Grone–Merris Conjecture [7], which was proved by Bai [8], we call it GMB, theorem says that the following upper bound is valid

$$\sum_{i=1}^{t} \mu_i(L(G)) \le \sum_{i=1}^{t} d_i^*(G). \tag{0.1}$$

It is known [9] that for threshold graphs there is equality in the last relation.

Say that a graph G on n nodes with $m = m(G)$ edges is spectrally threshold dominated [10] if for each $t \in [n]$ there is a threshold graph $\hat{G}$ having the same number of nodes and edges satisfying

$$\sum_{i=1}^{t} \mu_i(L(G)) \le \sum_{i=1}^{t} \mu_i(L(\hat{G})) = \sum_{i=1}^{t} d_i^*(L(\hat{G})).$$

In paper [10] Helmberg and Trevisan proved the following

Conjecture 1. *Graph G is spectrally threshold dominated iff Conjecture for this graph is valid.*

We introduce here their proof via construction of the set of conjugate degrees of optimal threshold graphs.

We construct for arbitrary $n, m = m(G), t$ threshold graph T that attains Brouwer's bound for the sum of eigenvalues. Denote by $\mathrm{Tr}(n, m)$ the set of threshold graphs with n vertices and m edges. To each graph with degree sequence $d_i \ge d_{i+1}$ define Ferrers diagram of n rows, s. t. the i-th row displays d_i boxes aligned to the left.

Next we demonstrate for arbitrary $t \in [n]$ that

$$\min\left\{tn, m(G) + t(t+1)/2, 2m(G)\right\} = \max_{T \in \mathrm{Tr}(n,m)} \sum_{i=1}^{t} d_i^*(T).$$

This together with (0.1) deliver the proof of Conjecture 1.

Depending on the relation between t, n and $m(G)$, we consider the following cases:

Case 1: $\min\{tn, m(G) + t(t+1)/2, 2m(G)\} = tn$. Consider the threshold graph T constructed by filling up the Ferrers diagram below the diagonal in column wise order (on and above the diagonal in corresponding row wise order). The first t columns below the diagonal are fully filled because they require $tn - t(t+1)/2 \le m(G)$ boxes. Hence T satisfies $d_i^*(T) = n$ for $i \in [t]$ and $\sum_{i=1}^{t} d_i^*(T) = tn$. This is the maximum attainable over all threshold graphs on n nodes.

Case 2. $\min\{tn, m(G) + t(t+1)/2, 2m(G)\} = m(G) + t(t+1)/2$. In this case put $h = \lfloor \frac{m(G)}{t} + \frac{t+1}{2} \rfloor < n$ and $r = m(G) + t(t+1) - th < t$. Note that this implies $h \ge t+1$. Define a threshold graph T on n nodes with $m(T) = m(G)$ edges of trace t by the conjugate degrees

$$d_i^*(T) = \begin{cases} h+1, & i \le r, \\ h, & r < i \le t, \end{cases}$$

then $\sum_{i=1}^{t} \lambda_i(T) = \sum_{i=1}^{t} d_i^*(T) = m(T) + t(t+1)/2$. This value cannot be exceeded by any threshold graph on n nodes with m edges by the GMB theorem, because in the Ferrers diagram of the conjugate degrees up to column t all boxes are used on and above the diagonal, while all possible m boxes are included below the diagonal.

Case 3. $\min\{tn, m(G)+t(t+1)/2, 2m(G)\} = 2m(G)$. Put $h = \max\{h \in [n] : h(h+1) \leq 2m(G)\} < t$ and $r = (2m(G) - h(h+1))/2 < h+1$, then the threshold graph T of trace h with conjugate degrees

$$d_i^*(T) = \begin{cases} h+2, & i \leq r, \\ h+1, & r < i \leq h, \\ r, & i = h+1, \\ 0, & h+1 < i \end{cases}$$

satisfies $\sum_i^t \lambda_i(T) = \sum_i^t d_i^*(T) = 2m(T)$ and this is the maximum attainable over all threshold graphs T with $m(T) = m(G)$ edges.

Define Laplacian energy of graph as follows

$$LE(G) = \sum_{i=1}^{n} \left| \mu_i(L(G)) - \frac{2m(G)}{n} \right|.$$

The main result of the paper [10] is the following

Theorem 0.1. *For each spectrally threshold dominated graph G there exists a threshold graph with the same number of nodes and edges whose Laplacian energy is at least as large as that of G.*

1. Preliminary remarks

Here we gather preliminary results that will be useful later.

Let $\bar{G} = \binom{[n]}{2} - G$ denote the complement of G. Then, [9]:

$$\mu_i(L(G)) = n - \mu_{n-i}(L(\bar{G})), \ \ i = 1, \ldots, n-1.$$

The following duality result will be key in our work. It follows directly from the proof of [2, Theorem 6], by including Δ 's with proper indices in the calculation.

Theorem 1.1 (see [11]). *For every graph G,*

$$\Delta_t(G) = \Delta_{n-t-1}(\bar{G}).$$

In particular, G satisfies BC_t if and only if $\bar{G}$ satisfies BC_{n-t-1}.

On the other hand, once G satisfy BC_t, the graph obtained by adding an isolated vertex, $G \cup \{v\}$, trivially satisfy BC_t. Then, from Theorem 1.1, we conclude that the graph $G' = \overline{\bar{G} \bigcup \{v\}}$ obtained by adding a dominating vertex v satisfies BC_{t+1} :

$$\Delta_{t+1}(G') = \Delta_{t+1}(\overline{\bar{G} \cup \{v\}}) = \Delta_{n-t-1}(\bar{G} \cup \{v\}) = \Delta_{n-t-1}(\bar{G}) = \Delta_t(G). \tag{1.1}$$

Given $G \subset \binom{[n]}{2}$, we define the *threshold family of* G, $\mathcal{T}(\mathcal{G})$, as the family of all graphs obtained from G by adding complete or empty vertices. Note that the family of threshold graphs defined in the Introduction coincides with $\mathcal{T}(\emptyset)$. From Theorem 1.1 and equality (1.1) we conclude that G satisfy Conjecture iff an element in $\mathcal{T}(\mathcal{G})$ does so. From this fact it follows

Lemma 1.1. *Brouwer's Conjecture is valid for every* n *and* t *provided that* $BC_{t'}$ *holds for every graph* G *with* n' *vertices where* $t' = \frac{n'-1}{2}$ *if* n' *is odd or* t' *equal to either* $\frac{n'-2}{2}$ *or* $\frac{n'}{2}$ *if* n' *is even.*

We call the explicit t's in Lemma 1.1 as the *middle* t*'s*. In what follows we will consider an inductive approach on n to prove that BC_t holds for the middle t's, whenever it holds for middle t's for graphs with fewer vertices. To this end, we remove one vertex of G and derive a special basis of R^n where explicit bounds can be inferred. Recall the following formula for $L(G)$:

$$(L(G)v, v) = \frac{1}{2} \sum_{(p,q)\in E} (v_p - v_q)^2.$$

We have [12, Cor 4.3.18]

$$\begin{aligned} S_t(G) &= \max \Big\{ \sum_{i=1}^{t} (L(G)x_i, x_i) \Big| x_1, \dots, x_t, (x_i, x_j) = \delta_{ij} \Big\}, \\ &= \max \big\{ \mathrm{tr}(L(G)_V) | V \text{ is a } t \text{ dimensional subspace of } R^n \big\} = \sum_{i=1}^{t} (L(G)z_i, z_i) \qquad (1.2) \end{aligned}$$

for $\{z_1, \dots, z_n\}$ an orthonormal set of eigenvectors corresponding to non-increasing eigenvalues of $L(G)$, and $z_n = z = (1/\sqrt{n}, \dots, 1/\sqrt{n})$.

From the last equality we conclude that

$$S_t(G) = \sum_{i=1}^{t} (L(G)x_i, x_i)$$

for any orthonormal basis $\{x_1, \dots, x_t\}$ of $\mathrm{span}\{z_1, \dots, z_t\}$.

We have

$$\begin{aligned} S_t(G) &= \max_{\{h_j,\ j\in[t]\}\in \mathrm{ort}(n,t)} \sum_{i=1}^{t} (L(G)x_i, x_i) \\ &\le \max_{\{h_j,\ j\in[t]\}\in \mathrm{ort}(n,t)} \sum_{i=1}^{t} (Dh_i, h_i) + \max_{\{h_j,\ j\in[t]\}\in \mathrm{ort}(n,t)} \sum_{i=1}^{t} (Ah_i, h_i) \\ &\le \sum_{i=1}^{t} d_i + \sqrt{t \sum_{i=1}^{n} \alpha_i^2} \le \sum_{i=1}^{t} d_i + \sqrt{nm} \le \sum_{i=1}^{t} d_i + n\sqrt{n}, \qquad (1.3) \end{aligned}$$

where α_i, $i \in [n]$ are eigenvalues of A and $\mathrm{ort}(n,t)$ is the family of sets of t orthonormal vectors in R^n.

Lemma 1.2. *There exists an orthonormal basis* $\{x_1, \dots, x_t\}$ *of span*$\{z_1, \dots, z_t\}$ *and orthonormal basis* $\{x_{t+1}, \dots, x_{n-1}\}$ *of span*$\{z_{t+1}, \dots, z_{n-1}\}$ *for any* $t \in [n-1]$ *such that* $x_i = (0, \dots, 0, x_{i,i}, x_{i,i+1}, \dots, x_{i,n})$, $i \in [t]$, $x_i = (0, \dots, 0, x_{i-t,i}, x_{i-t-1,i}, \dots, x_{n,i})$, $i \in [t+1, n-1]$.

We skip the usual proof of this Lemma, it contains the statement that one can choose the basis of such form in arbitrary subspace of dimension t and $n - t - 1$.

From now we fix a basis as in Lemma 1.2 and denote it by $\{x_1,\dots,x_t,x_{t+1},\dots,x_{n-1}\}$. It is easy to see that $0\le x_{1,1},x_{t+1,1}\le\sqrt{\frac{n-1}{n}}$. This is because $\sum_{i=1}^n x_{i,j}^2=1$ and $x_{n,j}=z_j=\frac{1}{\sqrt{n}}$. We further assume $0<x_{1,1},x_{t+1,1}<\sqrt{\frac{n-1}{n}}$, since the extremal cases are easily dealt with. The existence of x_1 also allows our induction step. Let $x_1,\dots,x_t$ be as in Lemma 1.2. Given $G\subset\binom{[n]}{2}$, consider the subgraph $G-\{1\}$ obtained by removing the first vertex of G, together with its edges.

We have

$$S_t(G)=\sum_{i=1}^t(x_i,L(G)x_i)=\sum_{i=2}^t(x_i,L(G-\{1\})x_i)+\sum_{p;(1,p)\in E}\sum_{i=2}^t x_{i,p}^2+(x_1,L(G)x_1)$$
$$\le S_{t-1}(G-\{1\})+\omega_1+(x_1,L(G)x_1),$$

where

$$\omega_1=\sum_{q:(1,q)\in E}\sum_{i=2}^t(x_{i,1}-x_{i,q})^2=\sum_{q:(1,q)\in E}\sum_{i=2}^t x_{i,q}^2\le d_1. \tag{1.4}$$

In particular, if $G-\{1\}$ satisfies BC_{t-1}, then G satisfies BC_t if

$$\omega_1+(x_1,L(G)x_1)\le t+d_1. \tag{1.5}$$

Equivalently, we can work with the complement graph, $\bar G$, and show that $\mathrm{BC}_{\bar t}$ holds if $\bar G-\{1\}$ satisfies $\mathrm{BC}_{\bar t-1}$ and

$$\bar\omega_1+(x_{t+1},L(\bar G)x_{t+1})\le\bar t+\bar d_1. \tag{1.6}$$

Here we take x_{t+1} as the only vector with (possibly) non-zero first coordinate, and

$$\bar t=n-1-t,\quad \bar d_1=n-1-d_1,\quad \bar\omega_1=\sum_{q:(1,q)\in\bar E}\sum_{i=t+2}^{n-1}(x_{i,1}-x_{i,q})^2=\sum_{q:(1,q)\in\bar E}\sum_{i=t+2}^{n-1}x_{i,q}^2\le\bar d_1. \tag{1.7}$$

It is easy to see that we can choose arbitrary $p\in[n]$ instead of the first coordinate in above consideration with substitution $1\leftrightarrow p$ in the formulas, we use this consideration below several times.

The key elements in the paper are the following bounds on $(L(G)x_1,x_1)$

P r o p o s i t i o n 1.1. *Let* x_1 *be as in Lemma* 1.2 *and* $x_{1,1}>0$. *Then,*

$$(x_1,L(G)x_1)\le\begin{cases} d_1+\sqrt{d_1\frac{1-x_{1,1}^2}{x_{1,1}^2}}, & x_{1,1}^2\ge\frac{d_1}{d_1+1};\\ \frac{nd_1}{n-1}+\sqrt{\frac{d_1\bar d_1}{n-1}\frac{1-\frac{n}{n-1}x_{1,1}^2}{x_{1,1}^2}}, & x_{1,1}^2<\frac{d_1}{d_1+1}.\end{cases} \tag{1.8}$$

Likewise,

$$(x_{t+1},L(\bar G)x_{t+1})\le\begin{cases} \bar d_1+\sqrt{\bar d_1\frac{1-x_{t+1,1}^2}{x_{t+1,1}^2}}, & x_{t+1,1}^2\ge\frac{\bar d_1}{\bar d_1+1};\\ \frac{n\bar d_1}{n-1}+\sqrt{\bar d_1\left(1-\frac{\bar d_1}{n-1}\right)\frac{1-\frac{n}{n-1}x_{t+1,1}^2}{x_{t+1,1}^2}}, & x_{t+1,1}^2<\frac{\bar d_1}{\bar d_1+1}.\end{cases}$$

P r o o f. By eventually replacing x_1 by $-x_1$, we assume that $x_{1,1} > 0$.
We have

$$(x_1, L(G)x_1)x_{1,1} = d_1 x_{1,1} - \sum_{p:(1,p)\in E} x_{t,p} \le d_1 x_{1,1} + \Big| \sum_{p:(1,p)\in E} x_{1,p} \Big|$$

and

$$\Big| \sum_{p:(1,p)\in E} x_{1,p} \Big| = \Big| \sum_{p:(1,p)\in \bar{E}} x_{1,p} + x_{1,1} \Big| \le x_{1,1} + \Big| \sum_{p:(1,p)\in \bar{E}} x_{1,p} \Big|.$$

Using Jensen inequality we obtain:

$$\Big| \sum_{p:(1,p)\in E} x_{1,p} \Big| \le \sqrt{d_1 x}, \qquad \Big| \sum_{p:(1,p)\in \bar{E}} x_{1,p} \Big| \le \sqrt{\bar{d}_1(1 - x_{1,1}^2 - x)},$$

where $x = \sum_{p:(1,p)\in E} x_{1,p}^2$.

Therefore,

$$\Big| \sum_{p:(1,p)\in E} x_{1,p} \Big| \le \max_{x\in[0,1-x_{1,1}^2]} \min \Big\{ \sqrt{d_1 x}, x_{1,1} + \sqrt{\bar{d}_1(1 - x_{1,1}^2 - x)} \Big\}$$

$$= \begin{cases} \sqrt{d_1(1 - x_{1,1}^2)}, & x_{1,1}^2 \ge \frac{d_1}{d_1+1}; \\ \frac{x_{1,1} d_1}{n-1} + \sqrt{\frac{d\bar{d}_1}{n-1}\big(1 - \frac{n}{n-1} x_{1,1}^2\big)}, & \text{otherwise.} \end{cases}$$

The first bound on $x_{1,1}^2$ is equivalent to

$$\sqrt{d_1(1 - x_{1,1}^2)} \le x_{1,1},$$

making $\sqrt{d_1(1 - x_{1,1}^2)}$ the solution to the $\max\min$ problem. Otherwise, since

$$x \mapsto \sqrt{\bar{d}_1(1 - x_{1,1}^2 - x)}$$

is decreasing, the $\max\min$ is achieved when

$$\sqrt{d_1 x} = x_{1,1} + \sqrt{\bar{d}_1(1 - x_{1,1}^2 - x)}.$$

We manipulate this equation as follows:

$$(\sqrt{d_1 x} - x_{1,1})^2 = \bar{d}_1(1 - x_{1,1}^2 - x) \;\Leftrightarrow\; (n-1)x - 2\sqrt{d_1} x_{1,1}\sqrt{x} + x_{t,1}^2 - \bar{d}_1(1 - x_{1,1}^2) = 0$$

$$\Leftrightarrow \sqrt{x} = \frac{\sqrt{d_1} x_{1,1}}{n-1} + \sqrt{\Big(\frac{\sqrt{d_1} x_{1,1}}{n-1}\Big)^2 - \frac{x_{1,1}^2 - \bar{d}_1(1 - x_{1,1}^2)}{n-1}}$$

$$= \frac{\sqrt{d_1} x_{1,1}}{n-1} + \sqrt{\frac{d_1 x_{1,1}^2 - (n-1)x_{1,1}^2 + (n-1)\bar{d}_1(1 - x_{1,1}^2)}{(n-1)^2}}$$

$$= \frac{\sqrt{d_1} x_{1,1}}{n-1} + \sqrt{\frac{\bar{d}_1(1 - x_{1,1}^2 - \frac{1}{n-1} x_{1,1}^2)}{n-1}} = \frac{\sqrt{d_1} x_{1,1}}{n-1} + \sqrt{\frac{\bar{d}_1}{n-1}\Big(1 - \frac{n}{n-1} x_{1,1}^2\Big)}.$$

The result is concluded by multiplying the last expression by $\sqrt{d_1}$. □

Before proceeding, we remark the following inequality that follows from the last proof.

Lemma 1.3. *Suppose* $x_{1,1}^2 < \frac{d_1}{d_1+1}$. *Then,* $\left|\sum_{p:(1,p)\in E} x_{1,p}\right| \le \sqrt{\frac{d_1\bar{d}_1}{n-1}}$.

P r o o f. In the proof of Proposition 1.1, we concluded that

$$\Big|\sum_{p:(1,p)\in E} x_{1,p}\Big| \le \frac{x_{1,1}d_1}{n-1} + \sqrt{\frac{d\bar{d}_1}{n-1}\Big(1-\frac{n}{n-1}x_{1,1}^2\Big)}.$$

Proof follows from the observation that r.h.s. of last inequality is decreasing function of $x_{1,1}$ and hence achieved its maximum for $x_{1,1} > 0$ at $x_{1,1} = 0$. □

An extra inequalities are also needed.

Recall that x_1, x_{t+1} are the only vectors in $\{x_1, \ldots, x_{n-1}\}$ with non-zero first coordinates. To motivate the next inequality, we also recall that the first vertex is complete if and only if the vector

$$z = \Big(\sqrt{\frac{n-1}{n}}, -\frac{1}{\sqrt{n(n-1)}}, \ldots, -\frac{1}{\sqrt{n(n-1)}}\Big)$$

is in the span of $\{x_1, \ldots, x_t\}$.

Next, we measure how much this vector does not belong to this t-subspace.

There exists $0 < \lambda < 1$ and a vector $y = (0, y_2 \ldots, y_n)$, $\sum_{p=2}^n y_p = 0$, $\sum_{p=2}^n y_p^2 = 1$ such that

$$\begin{aligned} x_1 &= z\sqrt{\lambda} + \sqrt{1-\lambda}y, \\ x_{t+1} &= z\sqrt{1-\lambda} - \sqrt{\lambda}y. \end{aligned}$$

Further denote:

$$\begin{aligned} B &= \frac{1}{2}(y, L(G)y) = \sum_{p<q,\ (p,q)\in E} (y_p - y_q)^2, \\ \bar{B} &= \frac{1}{2}(y, L(\bar{G})y) = \sum_{p<q,\ (p,q)\in \bar{E}} (y_p - y_q)^2. \end{aligned} \tag{1.9}$$

Then, using inequalities

$$\Big|\sum_{p:(1,p)\in E} y_p\Big| \le \max_{x\in[0,1]} \min\Big\{\sqrt{d_1 x}, \sqrt{\bar{d}_1(1-x)}\Big\}$$

we have

$$\begin{aligned} (x_1, L(G)x_1) &= \lambda d_1\frac{n}{n-1} + (1-\lambda)B - 2\sqrt{\lambda(1-\lambda)\frac{n}{n-1}}\sum_{p:(1,p)\in E} y_p \\ &\le \lambda d_1\frac{n}{n-1} + (1-\lambda)B + 2\sqrt{\frac{n}{n-1}\lambda(1-\lambda)d_1\Big(1-\frac{d_1}{n-1}\Big)}; \\ (x_{t+1}, L(\bar{G})x_{t+1}) &= (1-\lambda)\bar{d}_1\frac{n}{n-1} + \lambda\bar{B} - 2\sqrt{\lambda(1-\lambda)\frac{n}{n-1}}\sum_{p:(1,p)\in \bar{E}} y_p \\ &\le (1-\lambda)\bar{d}_1\frac{n}{n-1} + \lambda\bar{B} + 2\sqrt{\frac{n}{n-1}\lambda(1-\lambda)\bar{d}_1\Big(1-\frac{\bar{d}_1}{n-1}\Big)}. \end{aligned} \tag{1.10}$$

Optimization over λ deliver the following bounds

P r o p o s i t i o n 1.2. *Let x_t be as above. Then,*

$$(x_1, L(G)x_1) \le \frac{d_1\frac{n}{n-1}+B}{2} + \frac{1}{2}\sqrt{\Big(d_1\frac{n}{n-1}-B\Big)^2 + 4\frac{n}{n-1}d_1\Big(1-\frac{d_1}{n-1}\Big)};$$
$$(x_{t+1}, L(\bar{G})x_{t+1}) \le \frac{\bar{d}_1\frac{n}{n-1}+\bar{B}}{2} + \frac{1}{2}\sqrt{\Big(\bar{d}_1\frac{n}{n-1}-\bar{B}\Big)^2 + 4\frac{n}{n-1}\bar{d}_1\Big(1-\frac{\bar{d}_1}{n-1}\Big)}.$$

P r o o f. We maximize the expression in (1.10) for $0 < \lambda < 1$. To this aim, we analyze the derivative of the expression with respect to λ:

$$d_1\frac{n}{n-1} - B + \sqrt{\frac{d_1\bar{d}_1}{n-1}}\frac{1-2\lambda}{\sqrt{\lambda(1-\lambda)}}. \tag{1.11}$$

Observe that the derivative goes to $+\infty$ and $-\infty$ as λ goes to 0 and 1, respectively.

Therefore, we conclude that the maximum is in the interior. On the other hand setting expression (1.11) to zero gives:

$$\lambda^2 - \lambda + \frac{1}{4+A^2} = 0, \quad A = \frac{\big(d_1\frac{n}{n-1}-B\big)(n-1)}{\sqrt{d_1\bar{d}_1 n}}.$$

The maximum is achieved at:

$$\lambda_\pm = \frac{1}{2}\Big(1 \pm \frac{A}{\sqrt{4+A^2}}\Big).$$

The proof is concluded by replacing λ by $\lambda_\pm$ in (1.10), observing that $\lambda_\pm = 1-\lambda_\mp$. □

Using Proposition 1.2 and conditions (1.5), we conclude that if graph $G-\{1\}$ satisfies BC_{t-1} and

$$\frac{d_1\frac{n}{n-1}+B}{2} + \frac{1}{2}\sqrt{\Big(d_1\frac{n}{n-1}-B\Big)^2 + 4d_1\Big(1-\frac{d_1}{n-1}\Big)\frac{n}{n-1}} + \omega_1 \le t + d_1, \tag{1.12}$$

then graph G satisfies BC_t. Simular using condition (1.6)

$$\frac{\bar{d}_1\frac{n}{n-1}+\bar{B}}{2} + \frac{1}{2}\sqrt{\Big(\bar{d}_1\frac{n}{n-1}-\bar{B}\Big)^2 + 4\bar{d}_1\Big(1-\frac{\bar{d}_1}{n-1}\Big)\frac{n}{n-1}} + \bar{\omega}_1 \le \bar{t} + \bar{d}_1 \tag{1.13}$$

we conclude that if graph $\bar{G}-\{1\}$ satisfies $\mathrm{BC}_{\bar{t}-1}$, then graph $\bar{G}$ satisfies $\mathrm{BC}_{\bar{t}}$.

2. Proof of Conjecture

We describe the key steps in the proof.

First case. Using (1.8) and assuming condition $x_{1,1}^2 \ge \frac{d_1}{d_1+1}$ or $x_{t+1,1}^2 \ge \frac{\bar{d}_1}{\bar{d}_1+1}$ we make inductive proof BC_t for graph G or $\bar{G}$ assuming that BC_{t-1} for $G-\{1\}$ is valid.

Second case. We assume that there exists $p \in [n]$ s.t. $\omega_p \le t(1-\delta)$. Because we can make permutation $p \leftrightarrow 1$ vertices of graph in arbitrary way, w.l.o.g. we set $p=1$. Assume also

$$x_{1,1}^2 \ge \frac{2}{n\delta^2}, \quad \frac{1}{5} > \delta > 2n^{-1/3} \tag{2.1}$$

and $t-B>\frac{2}{\delta}$. At first we use inequality (1.5) which we reduce to (2.1) for one step inductive proof BC_t for graph G under the condition that BC_{t-1} for graph $G-\{1\}$ is true. Next we consider the case $B\geq t-\frac{2}{\delta}$. Then we come to contradiction to the condition (2.1).

Third case. $\omega_q>t(1-\delta),\ q\in[n]$. Two situations are possible.

When

$$m(\bar G)\geq\binom{t}{2}(1+3\delta),$$

we prove BC_{t-1} for graph $\bar G$ directly by using bound (1.2).

In the case

$$m(\bar G)<\binom{t}{2}(1+3\delta),$$

we first assume that there exist $p\in[n]$ s.t. $t(1+\delta)\geq d_p,\ \bar d_p\geq t(1-\delta)$. Next we show that there exist set $\mathcal R\subset[n]$ s.t. $\bar d_q\leq 7n\delta^{1/4},\ \ q\in\mathcal R$,

$$a=|\mathcal R|=\Big[n(1-8\delta^{1/4})\Big(1-\sqrt{1-\frac{1}{2(1-8\delta^{1/4})^2}}\Big)\Big].$$

By permutation of vertices of graph w.l.o.g. we can assume that $\mathcal R=[a]$. We make a steps of induction adding step by step $[a]$ vertices to the graph $G-[a]$ and for each step $i\in[a]$ we prove the BC_{t-i} for the graph $G-[i]$ under the assumption that BC_{t-i-1} is valid for graph $G-[i+1],\ i\in[a]$. Choice of a in (2.11) allows to prove BC_{t-a} directly by using bound (1.2).

In **the last case** assumption is $d_q>t(1+\delta)$ or $\bar d_q\leq(1-\delta),\ \ q\in[n],\ \ \mathrm{BC}_{t-1}$ for $\bar G$ is proved directly, using bound (1.2).

Next we use this scheme to demonstrate the proof in details.

We use induction on n to prove BC and assume that BC is true for $n\leq 10^{24}$. One can significantly improve this bound for n by following the proof in this article more carefully.

Let $\{x_i,\ i\in[n-1]\}$ be the set of eigenvectors of $L(G)$. Considering Grassmannian frame F with row set $\{x_i,\ i\in[t]\}$ and complement frame $\bar F$ with row set $\{x_i,\ i\in[t+1,n-1]\}$.

Note, that

$$x_{1,1}^2+x_{t+1,1}^2=\frac{n-1}{n}.$$

W.l.o.g. we can assume that $x_{1,1}\in\Big(0,\sqrt{(n-1)/n}\Big)$.

As a first step, we observe that the case $x_{1,1}^2\geq\frac{d_1}{d_1+1}$ (respectively, $x_{t+1,1}^2\geq\frac{\bar d_1}{\bar d_1+1}$), is easily discarded.

Lemma 2.1. *Suppose that either $x_{1,1}^2\geq\frac{d_1}{d_1+1}$ or $x_{t+1,1}^2\geq\frac{\bar d_1}{\bar d_1+1}$. Then, BC holds for G.*

P r o o f. To prove BC for n and $x_{1,1}^2\geq\frac{d_1}{d_1+1}$, assuming that it is true for $n-1$, it is sufficient to prove the inequality

$$d_1+\sqrt{d_1\frac{1-x_{1,1}^2}{x_{1,1}^2}}+\omega_1\leq d_1+t$$

or

$$\omega_1\leq t-\sqrt{d_1\frac{1-x_{1,1}^2}{x_{1,1}^2}}.$$

Last inequality is trivial, since

$$\sqrt{d_1 \frac{1-x_{1,1}^2}{x_{1,1}^2}} \le \sqrt{d_1 \frac{1-\frac{d_1}{d_1+1}}{\frac{d_1}{d_1+1}}} \le 1.$$

The same consideration proves BC when $x_{t+1,1}^2 \ge \frac{\bar{d}_1}{\bar{d}_1+1}$. □

Taking into account conditions 1.5, 1.6 and Proposition 1.1 together we conclude that BC_t holds for G if one of the following inequalities is true:

$$\frac{d_1}{n-1} + \sqrt{d_1\Big(1-\frac{d_1}{n-1}\Big)\frac{1-\frac{n}{n-1}x_{1,1}^2}{x_{1,1}^2}} + \omega_1 \le t; \tag{2.2}$$

$$\frac{\bar{d}_1}{n-1} + \sqrt{\bar{d}_1\Big(1-\frac{\bar{d}_1}{n-1}\Big)\frac{1-\frac{n}{n-1}x_{t+1,1}^2}{x_{t+1,1}^2}} + \bar{\omega}_1 \le \bar{t}. \tag{2.3}$$

For the remaining of the paper, we consider $t=\frac{n}{2}$ when $n=2t$ and $t=\frac{n-1}{2}$ when $n=2t+1$. Assume at first that $\omega_1 < t(1-\delta)$.

Then using (2.2) we obtain the inequality

$$d_1\Big(1-\frac{d_1}{n-1}\Big)\frac{1-\frac{n}{n-1}x_{1,1}^2}{x_{1,1}^2} \le (t\delta-1)^2$$

or

$$x_{1,1}^2 \ge \frac{s\bar{s}(n-1)}{s\bar{s}n+(t\delta-1)^2}, \quad s=\frac{d_1}{n-1}, \quad \bar{s}=1-s.$$

The last inequality is satisfied if

$$x_{1,1}^2 \ge \frac{2}{n\delta^2}, \quad \frac{1}{5} > \delta > 3n^{-1/3}. \tag{2.4}$$

It is left to consider the reverse condition:

$$x_{1,1}^2 < \frac{2}{n\delta^2}.$$

In this case, we have:

$$\begin{aligned} d_1 &< \omega_1 + \sum_{p:(1,p)\in E}(x_{1,p}-x_{1,1})^2 < d_1x_{1,1}^2 + 2\sqrt{d_1}x_{1,1} + 1 + \omega_1 \\ &< \frac{1}{\delta^2} + \frac{2}{\delta} + 1 + \omega_1 < \frac{2}{\delta^2} + \omega_1 \le t\Big(1-\frac{\delta}{4}\Big), \quad 3n^{-1/3} < \delta < 10^{-1}. \end{aligned} \tag{2.5}$$

Imposing condition (1.12) and using inequality $\omega_1 \le d_1$, we obtain stronger condition

$$\frac{d_1\frac{n}{n-1}+B}{2} + \frac{1}{2}\sqrt{\Big(d_1\frac{n}{n-1}-B\Big)^2 + 4d_1\Big(1-\frac{d_1}{n-1}\Big)\frac{n}{n-1}} + d_1 \le d_1 + t \tag{2.6}$$

to prove that graph G satisfies BC_t if $G-\{1\}$ satisfies BC_{t-1}.

Hence

$$\sqrt{\Big(d_1\frac{n}{n-1}-B\Big)^2 + 4d_1\Big(1-\frac{d_1}{n-1}\Big)\frac{n}{n-1}} \le 2t - d_1\frac{n}{n-1} - B.$$

Assume that $B \le t - \frac{2}{\delta}$, then

$$4d_1\Big(1-\frac{d_1}{n-1}\Big)\frac{n}{n-1} \le \Big(2t - d_1\frac{n}{n-1} - B\Big)^2 - \Big(d_1\frac{n}{n-1} - B\Big)^2.$$

Hence we need to prove inequality

$$d_1\left(1-\frac{d_1}{n-1}\right)\frac{n}{n-1} \le \left(t - d_1\frac{n}{n-1}\right)(t-B).$$

To satisfy last inequality it is sufficient to impose condition

$$d_1 < t\Big(1-\frac{\delta}{4}\Big), \tag{2.7}$$

when $\delta > 3n^{-1/3}$, $t - B \ge \frac{2}{\delta}$.

At last, if $B \ge t - \frac{2}{\delta}$, then $\bar B < n - t + \frac{2}{\delta} = \frac{n}{2} + \frac{2}{\delta}$, $\bar d_1 > n - 1 - t\left(1-\frac{\delta}{4}\right) \ge \frac{n-2}{2}\left(1+\frac{\delta}{4}\right)$.

From other side,

$$\begin{aligned}\bar B &\ge \sum_{q:(1,q)\in \bar E} (x_{t+1,1} - x_{t+1,p})^2 \ge \bar d_1 x_{t+1,1}^2 - 2|x_{t+1,1}|\sqrt{\bar d_1(1-x_{t+1,1}^2)}\\ &> \frac{n-2}{2}\Big(1+\frac{\delta}{4}\Big)\Big(\frac{n-1}{n} - \frac{2}{n\delta^2}\Big) - 2\sqrt{\Big(\frac{2}{n\delta^2}+\frac{1}{n}\Big)\frac{n}{2}\Big(1+\frac{\delta}{4}\Big)}\\ &\ge \frac{n}{2} + n\frac{\delta}{8} - \Big(2+\frac{\delta}{4}\Big)\Big(1+\frac{1}{\delta^2}\Big) - \frac{2}{\delta}\Big(1+\frac{\delta}{8}\Big) > \frac{n}{2} + \frac{2}{\delta}, \quad \text{where } \frac{1}{5} > \delta > 3n^{-1/3}.\end{aligned} \tag{2.8}$$

This contradiction complete the proof in the case that exists $p \in [n]$, s.t. $\omega_p \le t(1-\delta)$. Here in the third inequality we use relation

$$x_{t+1,1}^2 = \frac{n-1}{n} - x_{1,1}^2$$

and in the forth inequality in (2.8) we use the inequality

$$2\sqrt{\Big(\frac{2}{n\delta^2}+\frac{1}{n}\Big)\frac{n}{2}\Big(1+\frac{\delta}{4}\Big)} < \frac{2}{\delta}\Big(1+\frac{\delta}{8}\Big).$$

Next we assume that $\omega_q > t(1-\delta)$, $q \in [n]$. Then $d_q \ge \omega_q > t(1-\delta)$ and $\bar d_q \le t(1+\delta)$, $q \in [n]$. $\mathrm{BC}_{\bar t}$, $\bar t = t, t-1$ to be true for complement graph $\bar G$ it is sufficient to impose inequality

$$\sum_{q=t+1}^{n-1} (x_q, L(\bar G)x_q)) = \sum_{q=t+1}^{n-1} \mu_q(L(\bar G)) \le \sum_{q=t+1}^{n-1} \bar d_q + n\sqrt{n} \le t^2(1+\delta) + n\sqrt{n} \le m(\bar G) + \binom{\bar t}{2}.$$

Here we use bound (1.3).

From the last inequality it follows that $\mathrm{BC}_{\bar t}$ is true for $\bar G$ if the number of edges $m(\bar G)$ satisfies the inequality

$$m(\bar G) \ge \binom{t}{2}(1+3\delta).$$

Assume now that there exists $p \in [n]$ s.t. $t(1+\delta) > d_p$, $\bar d_p \ge t(1-\delta)$.

Taking into account that $\omega_p > t(1-\delta)$ we have

$$\sum_{q:(p,q)\in\bar{E}}\sum_{i=1}^{t} x_{i,q}^2 < t - t(1-\delta) = t\delta.$$

Assuming uniform distribution on the set $\{q : (p,q) \in \bar{E}\}$,

$$E\Big(\sum_{i=1}^{t} x_{i,q}^2\Big) \le \frac{t\delta}{\bar{d}_p} \le \frac{t\delta}{t(1-\delta)} = \frac{\delta}{1-\delta}.$$

Using Markov inequality $P(X > CE(X)) \le \frac{1}{C}$, $C > 0$ and choosing $C = \delta^{-1/2}$, we have

$$P\Big(\sum_{i=1}^{t} x_{i,q}^2 > \frac{\sqrt{\delta}}{1-\delta}\Big) < \sqrt{\delta}.$$

Hence

$$\sum_{i=1}^{t} x_{i,q}^2 < \frac{\sqrt{\delta}}{1-\delta},$$

where $q \in J$ for some set $J \subset \{q : (p,q) \in \bar{E}\}$, $|J| > (1-\sqrt{\delta})(1-\delta)t$. Note also that $\omega_q \le t$.

Hence for the arbitrary $q \in J$

$$\begin{aligned} d_q &\le \sum_{r:(r,q)\in E}\sum_{i=1}^{t}(x_{i,r}-x_{i,q})^2 \le \sum_{r:(r,q)\in E}\sum_{i=1}^{t} x_{i,q}^2 + \sum_{r:(r,q)\in E}\sum_{i=1}^{t} x_{i,r}^2 - 2\sum_{r:(r,q)\in E}\sum_{i=1}^{t} x_{i,q}x_{i,r} \\ &\le t + d_q\frac{\sqrt{\delta}}{1-\delta} + 2!\sum_{r:(r,q)\in E}\sqrt{\sum_{i=1}^{t} x_{i,q}^2}\sqrt{\sum_{i=1}^{t} x_{i,r}^2} \le t + d_q\frac{\sqrt{\delta}}{1-\delta} + 2d_q\sqrt{\frac{\sqrt{\delta}}{1-\delta}} \end{aligned}$$

or

$$d_q \le \frac{t}{1-\frac{\sqrt{\delta}}{1-\delta}-2\sqrt{\frac{\sqrt{\delta}}{1-\delta}}} \le t(1+3\delta^{1/4}), \quad q\in J, \quad \delta > 3n^{-1/3}. \tag{2.9}$$

Because $d_p > \omega_p > t(1-\delta) > t(1-4\delta^{1/4})$, $p \in [n]$, we have inequalities

$$t(1-4\delta^{1/4}) < d_p, \bar{d}_q < t(1+4\delta^{1/4}), \quad \omega_q > t(1-\delta), \quad q \in J.$$

Thus

$$\begin{aligned} \sum_{q\in[n]\setminus J}\bar{d}_q &\le 2m(\bar{G}) - (1-4\delta^{1/4})t^2(1-\delta)(1-\sqrt{\delta}) \\ &\le 2\binom{t}{2}(1+3\delta) - (1-4\delta^{1/4})t^2(1-\delta)(1-\sqrt{\delta}) < 3n^2\sqrt{\delta}. \end{aligned}$$

W.l.o.g. we can assume $|J| = [t(1-\delta)(1-\sqrt{\delta})]$. Assuming uniform distribution on the set $[n]\setminus J$, we have

$$E(\bar{d}_q) = \frac{\sum_{q\in[n]\setminus J}\bar{d}_q}{|[n]\setminus J|} < \frac{3n^2\sqrt{\delta}}{n-t(1-\delta)(1-\sqrt{\delta})} < 7n\sqrt{\delta}.$$

Using Markov inequality we have

$$P\left(\bar{d}_q \geq 7n\delta^{1/4}\right) < \delta^{1/4}.$$

Thus there exists set $I \subset [n]\backslash J$, $|I| > n - t(1-\delta^{1/4})(1-\sqrt{\delta})(1-\delta) > t(1+\delta^{1/4})$ s.t. $\bar{d}_q < 7n\delta^{1/4}$, $q \in I$ and hence $d_q \geq n(1-7\delta^{1/4})$, $q \in I$. W.l.o.g. we can assume that $I = \lfloor t(1+\delta^{1/4})\rfloor$ and it is sufficient here to assume that $7\delta^{1/4} \leq 1/10$.

Last inequality and inequality $3n^{-1/3} < \delta$, which is imposed in (2.4), (2.5), (2.8), (2.9) leads to the condition $3n^{-1/3} < \delta < (70)^{-4}$. Hence $3n^{-1/3} < (70)^{-4}$. The last inequality to be true it is sufficient to impose condition $n > 10^{24}$.

We consider a coordinates $\mathcal{R}$ from the set I. W.l.o.g. we can assume that $\mathcal{R} = [a]$, where

$$a = \left[n(1-8\delta^{1/4})\left(1-\sqrt{1-\frac{1}{2(1-8\delta^{1/4})^2}}\right)\right].$$

Justification of the choice of a we make later. When passing step $i \in [a-1]$ we renumber vertices of graph $G-[i-1]$ as follows $i \to i-1$ skipping first $i-1$ positions in graphs $G-[i-1]$ and $\bar{G}-[i-1]$. On this way we redefine $t(i) = t-i$ vectors $x_{t+1},\ldots,x_{n-1}$ as follows $x_{t+i} = (0,\ldots,0,x_{t+i,i},\ldots,x_{t+i,n}) \to \tilde{x}_{t+i} = (x_{t+i,1},\ldots,x_{t+i,n-i+1})$. Complement set of orthonormal vectors of length $n-i+1$ we denote $\tilde{x}_1,\ldots,\tilde{x}_t$.

○ **Starting point for the process, described below and implemented "a" times.**

Using inequality (2.3) on i-th step and taking into account the inequalities $\bar{\omega}_1 < \bar{d}_1 \leq 7n\delta^{1/4}$ and $t(i) = t-i$ we obtain the inequality

$$\bar{d}_1\left(1-\frac{\bar{d}_1}{n-i}\right)\frac{1-\frac{n-i+1}{n-i}\tilde{x}^2_{t+i,1}}{\tilde{x}^2_{t+i,1}} \leq \left(t-1-i-7n\delta^{1/4}\right)^2. \tag{2.10}$$

Because $i \leq a$, we relax bound (2.10) to

$$\tilde{x}^2_{t+i,1} \geq \frac{7n\delta^{1/4}}{\left(\frac{1}{14}n-7n\delta^{1/4}\right)^2} > \frac{7\cdot 14^2\delta^{1/4}}{n}.$$

Assume now, that the opposite inequality is valid $\tilde{x}^2_{t+i,1} < \frac{7\cdot 14^2\delta^{1/4}}{n}$. Then we repeat considerations starting from equation (2.6) for $\bar{d}_1(G-[i]) \to \bar{d}_1(G-[i-1]) \leq 7n\delta^{1/4}$, $t \to t-i$, $\bar{B}(G-[i]) \to \bar{B}(G-[i-1])$.

According the induction process, we impose the inequality (1.13)

$$\frac{\bar{d}_1\frac{n-i+1}{n-i}+\bar{B}}{2}+\frac{1}{2}\sqrt{\left(\bar{d}_1\frac{n-i+1}{n-i}-\bar{B}\right)^2+4\bar{d}_1\left(1-\frac{\bar{d}_1}{n-i}\right)\frac{n-i+1}{n-i}}+\bar{\omega}_1 \leq \bar{d}_1+t-i. \tag{2.11}$$

Assume that $\bar{B} \leq t-i-1$. Making transformations of the last inequality, and using inequality $\bar{\omega}_1 \leq \bar{d}_1$, we impose stronger inequality

$$\bar{d}_1\left(1-\frac{\bar{d}_1}{n-i}\right)\frac{n-i+1}{n-i} \leq (t-i-\bar{B})\left(t-i-\bar{d}_1\frac{n-i+1}{n-i}\right).$$

We strength last inequality and obtain the bound

$$8n\delta^{1/4} \le (t-i-\bar{B})(t-i-7n\delta^{1/4})$$

or

$$\bar{B} \le t-i-\frac{8n\delta^{1/4}}{t-i-7n\delta^{1/4}}.$$

Assume now the opposite inequality

$$\bar{B} > t-i-\frac{8n\delta^{1/4}}{t-i-7n\delta^{1/4}}$$

or

$$B \le t+1+\frac{8n\delta^{1/4}}{t-i-7n\delta^{1/4}}. \tag{2.12}$$

From other side we have $d_1 > n(1-7\delta^{1/4})$ and $\tilde{x}_{1,1}^2 > \frac{n-i}{n-i+1} - \frac{7\cdot 14^2\delta^{1/4}}{n} > 1-\frac{2}{n}$, $\delta < (70)^{-4}$.

At the end we show that inequality (2.12) could not be satisfied, and we come to contradiction:

$$\begin{aligned} B &\ge \sum_{p:(1,p)\in E(G-[i-1])} (\tilde{x}_{1,1}-\tilde{x}_{1,p})^2 > d_1\tilde{x}_{1,1}^2 - 2|\tilde{x}_{1,1}|\sqrt{d_1} \\ &> n(1-7\delta^{1/4})\Big(1-\frac{2}{n}\Big) - 2\sqrt{n(1-7\delta^{1/4})\Big(1-\frac{2}{n}\Big)} > \frac{3}{2}t. \end{aligned}$$

Last inequality contradict to inequality (2.12) for $n > 10^{24}$.

Next we make above proof procedure (from sign ◯) for graph $\bar{G}-[i-1]$ and set of orthonormal vectors $x_{t+i},\ldots,x_{n-1}$. Step by step deleting vertex $i\in I$ from $\bar{G}$ on i-th step and assuming by induction that BC_{t-1-i} is true for graph $\bar{G}-[i]$ and set of vectors $x_{t+i+1},\ldots,x_{n-1}$ of length $n-i=2t-i$ and as before proving that BC_{t-i} is true for graph $\bar{G}-[i-1]$ and set of vectors $x_{t+i},\ldots,x_{n-1}$.

We make a steps of this induction process and obtain from graph $\bar{G}-[a]$ and set of vectors $x_{t+a+1},\ldots,x_{n-1}$ of length $n-a=2t-a$, graph $\bar{G}$ and set of vectors $x_{t+1},\ldots,x_{n-1}$ of length n. The complement to graph $\bar{G}-[a]$ is graph $G-[a]$ and complement set of orthonormal vectors $x_1,\ldots,x_t$ of length $n-a$. The BC_{t-a-1} is true for $\bar{G}-[a]$ iff BC_t is true for graph $G-[a]$.

Remind that we assume that

$$m(\bar{G}) \le \binom{t}{2}(1+3\delta)$$

or

$$\binom{2t+1}{2} - \binom{t}{2} \ge m(G) \ge 3\binom{t}{2}(1-\delta).$$

BC_t for $G-[a]$ is obviously true if $m(\bar{G}-[a]) \le \binom{t+1}{2}$. Because

$$d_q > n-1-\bar{d}_q \ge n-1-7n\delta^{1/4} > n(1-8\delta^{1/4}),$$

we have

$$\begin{aligned} m(G-[a]) &\le m(G) - na(1-8\delta^{1/4}) + \binom{a}{2} \\ &\le \binom{2t+1}{2} - \binom{t}{2} - na(1-8\delta^{1/4}) + \binom{a}{2} \le \binom{t+1}{2}. \end{aligned} \tag{2.13}$$

The last condition to be true it is sufficient to impose condition

$$a = \Big[n(1-8\delta^{1/4})\Big(1-\sqrt{1-\frac{1}{2(1-8\delta^{1/4})^2}}\Big)\Big]. \tag{2.14}$$

Note that $|I| = [t(1+\delta^{1/4})] > a$.

Assume now that $d_q > t(1+\delta)$, $i \in [n]$. Then $\bar{d}_q \le t(1-\delta)$. BC_{t-1} is true if

$$\sum_{q=1}^{t-1} \mu_i(L(\bar{G})) \le \sum_{i=1}^{t-1} \bar{d}_q + \sqrt{2m(\bar{G})t} \le t(t-1)(1-\delta) + \sqrt{nm(\bar{G})}$$
$$\le t(t-1)(1-\delta) + n\sqrt{n} \le m(\bar{G}) + \frac{n(n-2)}{8},$$

which is true when $m(\bar{G}) > \binom{t}{2}$, otherwise BC_{t-1} is trivially true for graph $\bar{G}$. This completes the proof.

Note that this version of the article has the following additions compared to the preprint published at [13]:

1. The formula (2.12) in the preprint was refined for the number a (in this article it became the formula (2.14));

2. Before the formula (2.14), in the second inequality for (2.13), the refined estimate $d_q > t(1-\delta)$ for $d_q > n(1-7\delta^{1/4})$ is used instead of the weaker one which allows us to obtain a suitable estimate value for the number a;

3. According to the formulas on page 11 of the preprint, the terms $1/n$ have been removed: taking its contribution is redundant and does not need to be taken into account;

4. The estimate for d_1 before formula (2.5) in the preprint (the inequality (2.7) in this paper) is corrected to match the estimate after formula (2.4) in the preprint ((2.5) in this article), so that the subsequent evaluation of $\bar{B}$ is transparent.

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