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On coincidence points in (q_1, q_2)-quasimetric space

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Abstract. In this paper, we present a theorem on a coincidence point of mappings which extends the Arutyunov theorem. The original version of the Arutyunov theorem guaranteed the existence of a coincidence point for two mappings acting in metric spaces, one of which is α-covering and the other is β-Lipschitz, where $\alpha > \beta$. This theorem was then extended to mappings acting in (q_1, q_2)-quasimetric spaces. In this paper, the problem of the existence of a coincidence point is solved for mappings acting from a (q_1, q_2)-quasimetric space to a set equipped with a distance satisfying only the identity condition (the distance vanishes if and only if the points coincide). Under conditions similar to those of the Arutyunov theorem, the existence of a coincidence point is proved. In addition, the questions of convergence of sequences of coincidence points of mappings ψ_n, φ_n to the coincidence point ξ of mappings ψ, φ are investigated under the convergences $\psi_n(\xi) \to \psi(\xi)$, $\varphi_n(\xi) \to \varphi(\xi)$.

Keywords: coincidence points, metric space, (q_1, q_2)-quasimetric space, covering mapping, Lipschitz mapping

Mathematics Subject Classification: 54H25, 54E40.

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НАУЧНАЯ СТАТЬЯ

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О точках совпадения в (q_1, q_2)-квазиметрическом пространстве

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Аннотация. В данной работе мы представляем теорему о точке совпадения отображений, которая обобщает теорему Арутюнова. В первоначальном варианте теоремы Арутюнова гарантируется существование точки совпадения двух отображений, действующих в метрических пространствах, одно из которых является α-накрывающим, а другое — β-липшицевым, причем $\alpha > \beta$. Затем эта теорема была распространена на отображения, действующие в (q_1, q_2)-квазиметрических пространствах. В данной статье задача о существовании точки совпадения решается для отображений, действующих из (q_1, q_2)-квазиметрического пространства в множество, снабженное расстоянием, удовлетворяющим лишь условию тождества (расстояние обращается в ноль тогда и только тогда, когда точки совпадают). При условиях, аналогичных предположениям теоремы Арутюнова, доказано существование точки совпадения. Кроме того, исследованы вопросы сходимости последовательностей точек совпадения отображений ψ_n, φ_n к точке совпадения ξ отображений ψ, φ при сходимости $\psi_n(\xi) \to \psi(\xi)$, $\varphi_n(\xi) \to \varphi(\xi)$.

Ключевые слова: точки совпадения, метрическое пространство, (q_1, q_2)-квазиметрическое пространство, накрывающее отображение, липшицево отображение

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Introduction

The study of coincidence points for mappings defined on metric and generalized metric structures has received increasing attention in recent years. The applications of this theory were in many fields, for example, integral equations [1, 2], differential equations [1, 3], and control problems [3–5]. In [6], the coincidence points of two mappings acting from one metric space to another metric space are investigated, then in [7], the coincidence points of two mappings acting from one (q_1, q_2)-quasimetric space to another (q_1, q_2)-quasimetric space are investigated, and in [8], the coincidence points of two mappings acting from a b-metric space to a space equipped with a distance satisfying only the identity condition are investigated. In all these researches, one of the mappings is closed and α-covering and the other is β-Lipschitz. In [9], E. S. Zhukovskiy formulated the notion of a geometric progression within the framework of generalized distance spaces and utilized it to determine sufficient conditions for the existence of coincidence points of mappings. Furthermore, works [10–12] address the problem of Lipschitz perturbations of covering mappings acting from a metric space into a space equipped with a distance that satisfies only the identity axiom.

In our study, we consider two mappings from a (q_1, q_2)-quasimetric space into a space equipped with a distance satisfying only the identity condition. We obtain conditions for the existence of coincidence points and their stability under changes in the mappings under consideration. These results are an extension of [6–8, 13].

1. Main concepts

Now, we introduce definitions of certain generalizations of the concepts of a distance and a metric space, which are required for the purposes of the present study (for more details, see [7–9]).

D e f i n i t i o n 1.1. Let X be a nonempty set. A *metric* on X is any function $\rho_X : X \times X \to \mathbb{R}_+$ that satisfies the following properties:

$$\rho_X(x, y) = 0 \;\Leftrightarrow\; x = y \quad \forall x, y \in X \quad \text{(Identity relation)}, \tag{1.1}$$

$$\rho_X(x, y) = \rho_X(y, x), \quad \forall x, y \in X \quad \text{(Symmetry)}, \tag{1.2}$$

$$\rho_X(x, z) \le \rho_X(x, y) + \rho_X(y, z), \quad \forall x, y, z \in X \text{ (Triangle inequality)}. \tag{1.3}$$

The pair (X, ρ_X) is called a *metric space.*

D e f i n i t i o n 1.2. Let X be a nonempty set, $\rho_X : X \times X \to \mathbb{R}_+$ be a function and $s \ge 1$ be a number. If (1.1) and (1.2) hold and the following inequality satisfied:

$$\rho_X(x, z) \le s(\rho_X(x, y) + \rho_X(y, z)) \quad \forall x, y, z \in X, \tag{1.4}$$

then ρ_X is called a *b-metric.* The pair (X, ρ_X) is called a *b-metric space.*

D e f i n i t i o n 1.3. Let X be a nonempty set, $\rho_X : X \times X \to \mathbb{R}_+$ be a function, $q_1 \ge 1$, and $q_2 \ge 1$ be numbers. If (1.1) holds and the following inequality satisfied:

$$\rho_X(x, z) \le q_1\rho_X(x, y) + q_2\rho_X(y, z), \qquad \forall x, y, z \in X, \tag{1.5}$$

then ρ_X is called a *(q_1, q_2)-quasimetric.* The pair (X, ρ_X) is called a *(q_1, q_2)-quasimetric space.* If additionally $q_1 = q_2 = 1$, then ρ_X is called a *quasimetric*, and (X, ρ_X) is a *quasimetric space.*

R e m a r k 1.1. If in Definition 1.3 $q_1 = q_2 = 1$ and (1.2) holds, then ρ_X is a metric, and (X, ρ_X) is a metric space.

In the following definitions, we consider additional notions for a (q_1, q_2)-quasimetric space.

D e f i n i t i o n 1.4. Let (X, ρ_X) be a (q_1, q_2)-quasimetric space and $q_0 \geq 1$. If

$$\rho_X(x, y) \leq q_0 \rho_X(y, x)$$

for all $x, y \in X$, then ρ_X is called a q_0-*symmetric* (q_1, q_2)-*quasimetric*, and the pair (X, ρ_X) is a q_0-*symmetric* (q_1, q_2)-*quasimetric space*. If, in addition, $q_0 = 1$, then (X, ρ_X) is called *a symmetric* (q_1, q_2)-*quasimetric space*.

R e m a r k 1.2. If in Definition 1.4 $q_0 = q_1 = q_2 = 1$, then (X, ρ_X) is an ordinary metric space.

D e f i n i t i o n 1.5. Let (X, ρ_X) be a (q_1, q_2)-quasimetric space. It is called a *weakly symmetric* (q_1, q_2)-*quasimetric space* if for any $\{x_i\} \subset X$, $x \in X$ we have

$$\lim_{i \to \infty} \rho_X(x, x_i) = 0 \quad \Rightarrow \quad \lim_{i \to \infty} \rho_X(x_i, x) = 0.$$

D e f i n i t i o n 1.6. A sequence of points $\{x_n\}$ in a (q_1, q_2)-quasimetric space (X, ρ_X) is said to *converge to a point* $x \in X$ if $\rho_X(x, x_n) \to 0$, $n \to \infty$. In this case, x is called a *limit* of this sequence. A sequence $\{x_n\}$ in a (q_1, q_2)-quasimetric space (X, ρ_X) is called a *fundamental sequence* or a *Cauchy sequence* if, for every $\varepsilon > 0$ there is an N such that $\rho_X(x_m, x_n) < \varepsilon$, for all $n > m > N$. A (q_1, q_2)-quasimetric space (X, ρ_X) is said to be *complete* if each of its fundamental sequences has a limit (possibly non-unique).

D e f i n i t i o n 1.7. Let Y be a nonempty set. We define a function $d : Y \times Y \to \mathbb{R}_+$ that satisfies only the identity axiom:

$$d(y_1, y_2) = 0 \;\Leftrightarrow\; y_1 = y_2 \quad \forall y_1, y_2 \in Y.$$

In this case, d is called a *distance*. A sequence $\{y_i\}$ in this space is said to *converge to* y if $d(y, y_i) \to 0$.

Here we indicate that the distance d on the set Y satisfies only the first metric condition, while the second (symmetry) and third (triangle inequality) conditions are not necessarily satisfied.

Also note that for a sequence $\{y_i\}$ and an element y in Y, if $d(y, y_i)$ converges to 0, this does not mean that $d(y_i, y)$ also converges to 0, and the limit may not be unique.

From [6], we provide extensions of the definitions of α-covering and β-Lipschitzness for mappings defined from (q_1, q_2)-quasimetric space (X, ρ_X) to a space (Y, d) equipped with a distance satisfying only the identity axiom.

D e f i n i t i o n 1.8. Let $\alpha > 0$. A mapping $\psi : X \to Y$ is called α-*covering* if the following relation is satisfied:

$$\forall x_0 \in X \;\; \forall y \in Y \;\; \exists x \in X \;:\; \psi(x) = y,$$

and

$$\rho_X(x, x_0) \leq \frac{1}{\alpha} d(\psi(x), \psi(x_0)).$$

D e f i n i t i o n 1.9. Let $\beta \geq 0$. A mapping $\varphi : X \to Y$ is called β-*Lipschitz* if the following relation is satisfied:

$$\forall x, x_0 \in X \quad d(\varphi(x), \varphi(x_0)) \leq \beta \rho_X(x, x_0).$$

For mappings acting from a (q_1, q_2)-quasimetric space X to a space with distance Y, the following standard definitions are used. A mapping $f : X \to Y$ is called *continuous at a point* $x \in X$ if for any sequence $\{x_n\} \subset X$ converging to x (i. e., $\rho_X(x, x_n) \to 0$), we have $f(x_n) \to f(x)$ (i. e., $d(f(x), f(x_n)) \to 0$). A mapping $f : X \to Y$ is called *closed at a point* $x \in X$ if the convergence of a sequence $\{x_n\} \subset X$ to x and the existence of $y \in Y$ such that $f(x_n) \to y$ imply the identity $f(x) = y$. A mapping continuous (closed) at all points is called *continuous* (*closed*). We stress that in contrast to metric spaces, the continuity of a mapping does not imply its closedness (see Example 1.1 from [14]). Also, we note that for the continuity of the mapping f at a point x, the uniqueness of the limit $f(x)$ of the sequence $\{f(x_i)\}$ is not required (see Example 1.1 from [15]).

D e f i n i t i o n 1.10. A point $x \in X$ is called a *coincidence point* of the mappings $\psi : X \to Y$ and $\varphi : X \to Y$ if the following relation is satisfied:

$$\psi(x) = \varphi(x).$$

The following theorem is the principal theorem of coincidence point in a metric space.

Theorem 1.1 (see [6]). *Let (X, ρ_X) be a complete metric space, (Y, ρ_Y) be a metric space, and the following conditions be satisfied: the mapping $\psi : X \to Y$ is α-covering and closed, the mapping $\varphi : X \to Y$ is β-Lipschitz, with $\alpha > \beta$. Then, for every $x_0 \in X$, there exists a coincidence point ξ of ψ and φ such that*

$$\forall x_0 \in X \quad \exists \xi \in X \, : \, \psi(\xi) = \varphi(\xi),$$

and

$$\rho_X(\xi, x_0) \leq \frac{1}{\alpha - \beta} \rho_Y(\varphi(x_0), \psi(x_0)).$$

2. Main results

In this section, consider mappings $\psi, \varphi : X \to Y$, where X is a (q_1, q_2)-quasimetric space such that $q_1 \geq 1$ and $q_2 \geq 1$, and Y is a space equipped with a distance satisfying only the identity axiom.

Let $\alpha > \beta \geq 0$ be given real numbers.

For $\theta \in \mathbb{R}$ and for $n \in \mathbb{N}$, we denote by $S(\theta, n)$ the sum of the first n terms of a geometric sequence $\sum_{i=0}^{n-1} \theta^i$, and thus,

$$S(\theta, n) = \frac{1 - \theta^n}{1 - \theta}.$$

Assume that $S(\theta, 0) = 0$ and $\beta^0 = 1$ when $\beta = 0$. For arbitrary values $q_0, q_1, q_2 \geq 1$, we set

$$m_0 = \min\{j \in \mathbb{N} \, : \, q_2 \beta^j < \alpha^j\}.$$

Note that the existence of m_0 follows from the assumption $\beta < \alpha$.

Under the assumption that $q_0^2\beta < \alpha$, we set

$$n_0 = \min\{j \in \mathbb{N} \,:\, q_1(q_0^2\beta)^j < \alpha^j\}.$$

The following theorem is the main result of this paper. Before stating it, we note that this theorem is practically identical to Theorem 4.5 in [7]. In Theorem 4.5, Y is a $(q_1^{'}, q_2^{'})$-quasimetric space (where $q_1^{'} \geq 1, q_2^{'} \geq 1$), but in the proposed theorem, Y is a space equipped with a distance satisfying only the identity axiom. Moreover, since every (q_1, q_2)-quasimetric space is a b-metric space, the proposed theorem is a general case of the result obtained in [8].

Theorem 2.1. *Let (X, ρ_X) be a complete (q_1, q_2)-quasimetric space and (Y, d) be a distance space. Suppose that the mapping $\psi : X \to Y$ is α-covering and closed, and the mapping $\varphi : X \to Y$ is β-Lipschitz. Fix any point $x_0 \in X$. Then the mappings ψ and φ have a coincidence point ξ for which the following estimate holds:*

$$\varliminf_{\eta\to\xi}\rho_X(x_0, \eta) \leq \frac{q_1^2\alpha^{m_0-1}S(q_2\beta/\alpha, m_0-1) + q_1(q_2\beta)^{m_0-1}}{\alpha^{m_0} - q_2\beta^{m_0}} d(\psi(x_0), \varphi(x_0)). \tag{2.1}$$

If the space (X, ρ_X) is weakly symmetric, then for ξ the following estimate is also verified:

$$\rho_X(x_0, \xi) \leq \frac{q_1^3\alpha^{m_0-1}S(q_2\beta/\alpha, m_0-1) + q_1^2(q_2\beta)^{m_0-1}}{\alpha^{m_0} - q_2\beta^{m_0}} d(\psi(x_0), \varphi(x_0)). \tag{2.2}$$

If the space (X, ρ_X) is q_0-symmetric and $q_0^2\beta < \alpha$, then for ξ the following estimates are also verified:

$$\rho_X(\xi, x_0) \leq q_0 q_2^2 \frac{q_2\alpha^{n_0-1}S(q_1q_0^2\beta/\alpha, n_0-1) + (q_1q_0^2\beta)^{n_0-1}}{\alpha^{n_0} - q_1(q_0^2\beta)^{n_0}} d(\psi(x_0), \varphi(x_0)), \tag{2.3}$$

$$\varliminf_{\eta\to\xi}\rho_X(\eta, x_0) \leq q_0 q_2 \frac{q_2\alpha^{n_0-1}S(q_1q_0^2\beta/\alpha, n_0-1) + (q_1q_0^2\beta)^{n_0-1}}{\alpha^{n_0} - q_1(q_0^2\beta)^{n_0}} d(\psi(x_0), \varphi(x_0)). \tag{2.4}$$

P r o o f. Let x_0 belong to X. If $\psi(x_0) = \varphi(x_0)$, then the theorem is verified. Now, if $\psi(x_0) \neq \varphi(x_0)$, we consider the sequence $\{x_i\} \subset X$. Let

$$a = d(\psi(x_0), \varphi(x_0)).$$

Since ψ is α-covering, then there exists a point $x_1 \in X$ such that:

$$\psi(x_1) = \varphi(x_0), \qquad \rho_X(x_0, x_1) \leq \frac{1}{\alpha} d(\psi(x_0), \varphi(x_0)) = \frac{1}{\alpha}\, a.$$

And as the mapping φ is β-Lipschitz, so

$$d(\varphi(x_0), \varphi(x_1)) \leq \beta\rho_X(x_0, x_1).$$

For the general case $i \in N$, we have:

$$\begin{gathered} \psi(x_i) = \varphi(x_{i-1}), \\ \rho_X(x_{i-1}, x_i) \leq \frac{1}{\alpha} d(\psi(x_{i-1}), \varphi(x_{i-1})) \leq \frac{1}{\alpha} d(\varphi(x_{i-2}), \varphi(x_{i-1})) \leq \frac{\beta}{\alpha}\rho_X(x_{i-2}, x_{i-1}). \end{gathered} \tag{2.5}$$

Indeed, suppose that the terms of the sequence $x_0, x_1, \dots, x_i$ satisfy the relations (2.5) and are constructed. Then there exists a point $x_{i+1} \in X$ such that:

$$\begin{gathered} \psi(x_{i+1}) = \varphi(x_i), \\ \rho_X(x_i, x_{i+1}) \le \frac{1}{\alpha} d(\psi(x_i), \varphi(x_i)) = \frac{1}{\alpha} d(\varphi(x_{i-1}), \varphi(x_i)) \le \frac{\beta}{\alpha} \rho_X(x_{i-1}, x_i). \end{gathered} \tag{2.6}$$

Thus, the construction of the sequence $\{x_i\} \subset X$ is completed. Using (2.6), we obtain:

$$\begin{aligned} \rho_X(x_i, x_{i+1}) \le \frac{\beta}{\alpha} \rho_X(x_{i-1}, x_i) \le \frac{\beta^2}{\alpha^2} \rho_X(x_{i-2}, x_{i-1}) \le \cdots \\ \le \frac{\beta^i}{\alpha^i} \rho_X(x_0, x_1) \le \frac{\beta^i}{\alpha^{i+1}} d(\psi(x_0), \varphi(x_0)) = \frac{\beta^i}{\alpha^{i+1}}\, a. \end{aligned} \tag{2.7}$$

Now we show that the sequence $\{x_i\}$ is fundamental. For all integers $i, j \ge 0$, by using the relations (2.7) and (1.5), we obtain:

$$\begin{aligned} \rho_X(x_i, x_{i+j}) &\le q_1 \rho_X(x_i, x_{i+1}) + q_2 \rho_X(x_{i+1}, x_{i+j}) \\ &\le q_1 \frac{\beta^i}{\alpha^{i+1}} a + q_2 \big(q_1 \rho_X(x_{i+1}, x_{i+2}) + q_2 \rho_X(x_{i+2}, x_{i+j}) \big) \\ &\le q_1 \frac{\beta^i}{\alpha^{i+1}} a + q_2 q_1 \rho_X(x_{i+1}, x_{i+2}) + q_2^2 \rho_X(x_{i+2}, x_{i+j}) \\ &\le q_1 \frac{\beta^i}{\alpha^{i+1}} a + q_2 q_1 \frac{\beta^{i+1}}{\alpha^{i+2}} a + q_2^2 \big(q_1 \rho_X(x_{i+2}, x_{i+3}) + q_2 \rho_X(x_{i+3}, x_{i+j}) \big) \\ &\le q_1 \frac{\beta^i}{\alpha^{i+1}} a + q_2 q_1 \frac{\beta^{i+1}}{\alpha^{i+2}} a + q_2^2 q_1 \rho_X(x_{i+2}, x_{i+3}) + q_2^3 \rho_X(x_{i+3}, x_{i+j}), \end{aligned}$$

etc. As the result we get

$$\begin{aligned} \rho_X(x_i, x_{i+j}) &\le q_1 \frac{\beta^i}{\alpha^{i+1}} a + q_2 q_1 \frac{\beta^{i+1}}{\alpha^{i+2}} a + q_2^2 q_1 \frac{\beta^{i+2}}{\alpha^{i+3}} a + \cdots + q_2^{j-2} q_1 \frac{\beta^{i+j-2}}{\alpha^{i+j-1}} a + q_2^{j-1} \frac{\beta^{i+j-1}}{\alpha^{i+j}} a \\ &\le q_1 \frac{\beta^i}{\alpha^{i+1}} a \left(1 + q_2 \frac{\beta^1}{\alpha^1} + q_2^2 \frac{\beta^2}{\alpha^2} + \cdots + q_2^{j-2} \frac{\beta^{j-2}}{\alpha^{j-2}} + q_1^{-1} q_2^{j-1} \frac{\beta^{j-1}}{\alpha^{j-1}} \right) \\ &= q_1 \frac{\beta^i}{\alpha^{i+1}} a \tilde{S}(j), \end{aligned}$$

where

$$\tilde{S}(j) = S(q_2 \frac{\beta}{\alpha}, j-1) + q_1^{-1} q_2^{j-1} \frac{\beta^{j-1}}{\alpha^{j-1}}, \quad j \in N, \quad \tilde{S}(0) = 0.$$

By using the latter, for all non-negative integers i and k, we obtain:

$$\begin{aligned} \rho_X(x_i, x_{i+k}) &\le q_1 \rho_X(x_i, x_{i+m_0}) + q_2 \rho_X(x_{i+m_0}, x_{i+k}) \\ &\le q_1 \rho_X(x_i, x_{i+m_0}) + q_2 \big(q_1 \rho_X(x_{i+m_0}, x_{i+2m_0}) + q_2 \rho_X(x_{i+2m_0}, x_{i+k}) \big) \\ &\le q_1 \rho_X(x_i, x_{i+m_0}) + q_2 q_1 \rho_X(x_{i+m_0}, x_{i+2m_0}) + q_2^2 \rho_X(x_{i+2m_0}, x_{i+k}) \\ &\le q_1 \rho_X(x_i, x_{i+m_0}) + q_2 q_1 \rho_X(x_{i+m_0}, x_{i+2m_0}) + q_2^2 \big(q_1 \rho_X(x_{i+2m_0}, x_{i+3m_0}) \\ &\qquad + q_2 \rho_X(x_{i+3m_0}, x_{i+k}) \big) \\ &\le q_1 \rho_X(x_i, x_{i+m_0}) + q_2 q_1 \rho_X(x_{i+m_0}, x_{i+2m_0}) + q_2^2 q_1 \rho_X(x_{i+2m_0}, x_{i+3m_0}) \\ &\qquad + q_2^3 \rho_X(x_{i+3m_0}, x_{i+k}). \end{aligned}$$

Continuing similar calculations, we obtain

$$\rho_X(x_i, x_{i+k}) \le q_1\rho_X(x_i, x_{i+m_0}) + q_2q_1\rho_X(x_{i+m_0}, x_{i+2m_0}) + q_2^2q_1\rho_X(x_{i+2m_0}, x_{i+3m_0}) + \cdots \\ + q_1q_2^{r-1}\rho_X(x_{i+(r-1)m_0}, x_{i+rm_0}) + q_2^r\rho_X(x_{i+rm_0}, x_{i+k}),$$

where $r = [k/m_0]$.

From the obtained inequality, it directly follows that

$$\begin{aligned}\rho_X(x_i, x_{i+k}) &\le q_1^2\frac{\beta^i}{\alpha^{i+1}}a\tilde{S}(m_0) + q_1^2q_2a\frac{\beta^{i+m_0}}{\alpha^{i+1+m_0}}\tilde{S}(m_0) + q_1^2q_2^2a\frac{\beta^{i+2m_0}}{\alpha^{i+1+2m_0}}\tilde{S}(m_0) + \cdots \\ &\quad + q_1^2q_2^{r-1}a\frac{\beta^{i+(r-1)m_0}}{\alpha^{i+(r-1)m_0+1}}\tilde{S}(m_0) + q_1q_2^ra\frac{\beta^{i+rm_0}}{\alpha^{i+rm_0+1}}\tilde{S}(k-rm_0) \\ &\le q_1^2\frac{\beta^i}{\alpha^{i+1}}a\tilde{S}(m_0)\Big(1 + q_2\frac{\beta^{m_0}}{\alpha^{m_0}} + q_2^2\frac{\beta^{2m_0}}{\alpha^{2m_0}} + \ldots + q_2^{r-1}\frac{\beta^{(r-1)m_0}}{\alpha^{(r-1)m_0}}\Big) \\ &\quad + q_1q_2^ra\frac{\beta^{i+rm_0}}{\alpha^{i+rm_0+1}}\tilde{S}(k-rm_0) \\ &\le q_1^2\frac{\beta^i}{\alpha^{i+1}}a\tilde{S}(m_0)S(q_2\frac{\beta^{m_0}}{\alpha^{m_0}}, r) + q_1q_2^ra\frac{\beta^{i+rm_0}}{\alpha^{i+rm_0+1}}\tilde{S}(k-rm_0),\end{aligned}$$

where $r = [k/m_0]$.

Therefore, by construction $q_2\frac{\beta^{m_0}}{\alpha^{m_0}} < 1$, we have:

$$\begin{aligned}\rho_X(x_i, x_{i+k}) &\le q_1^2\frac{\beta^i}{\alpha^{i+1}}a\tilde{S}(m_0)S(q_2\frac{\beta^{m_0}}{\alpha^{m_0}}, r) + q_1q_2^ra\frac{\beta^{i+rm_0}}{\alpha^{i+rm_0+1}}\tilde{S}(k-rm_0) \\ &\le q_1^2\frac{\beta^i}{\alpha^{i+1}}a\Big(\tilde{S}(m_0)S\big(q_2\frac{\beta^{m_0}}{\alpha^{m_0}}, r\big) + q_1^{-1}q_2^r\frac{\beta^{rm_0}}{\alpha^{rm_0}}\tilde{S}(k-rm_0)\Big) \\ &\le q_1^2\frac{\beta^i}{\alpha^{i+1}}a\Big(\frac{\tilde{S}(m_0)}{1-q_2\frac{\beta^{m_0}}{\alpha^{m_0}}} + q_1^{-1}\tilde{S}(k-rm_0)\Big).\end{aligned} \tag{2.8}$$

But the term $q_1^{-1}\tilde{S}(k-rm_0)$ is uniformly bounded for all k, $r = [k/m_0]$, because $0 \le k - rm_0 < m_0$.

Thus, the sequence $\{x_i\}$ is fundamental in complete space, so it has a limit ξ satisfying the following relation:

$$\lim_{i\to\infty} x_i = \xi \;\Leftrightarrow\; \lim_{i\to\infty}\rho_X(\xi, x_i) = 0. \tag{2.9}$$

By taking the limit in equality (2.6) and using the continuity of the mapping φ and the closedness of the mapping ψ, we obtain the equality

$$\varphi(\xi) = \psi(\xi).$$

If the number $k = lm_0$, $l \in \mathbb{N}$, then by construction,

$$\tilde{S}(k - rm_0) = 0.$$

Thus, for any natural number k that is a multiple of m_0, in equation (2.8) with $i = 0$, we have:

$$\rho_X(x_0, x_k) \le \frac{a}{\alpha}\Big(\frac{q_1^2\tilde{S}(m_0)}{1-q_2\frac{\beta^{m_0}}{\alpha^{m_0}}}\Big). \tag{2.10}$$

From this inequality, using (2.6) and (2.9), we obtain the corresponding estimate (2.1).

Now, suppose that the space (X, ρ_X) is weakly symmetric.

Using (2.6), (2.9), and (2.10), we obtain:

$$\begin{aligned}\rho_X(x_0,\xi) \leq & q_1\rho_X(x_0,x_k) + q_2\rho_X(x_k,\xi) \leq \\ & \leq q_1\frac{a}{\alpha}\left(\frac{q_1^2\tilde{S}(m_0)}{1-q_2\frac{\beta^{m_0}}{\alpha^{m_0}}}\right) \leq \frac{1}{\alpha}\frac{q_1^3\tilde{S}(m_0)}{1-q_2\frac{\beta^{m_0}}{\alpha^{m_0}}}d(\psi(x_0)),\varphi(x_0)).\end{aligned}$$

Thus, (2.2) is proven.

Now, we will prove (2.3) and (2.4).

Suppose the space (X, ρ_X) is q_0-symmetric, $q_0^2\frac{\beta}{\alpha} < 1$, and by using (2.7), we have

$$\frac{\rho_X(x_{i+1},x_i)}{q_0} \leq \rho_X(x_i,x_{i+1}) \leq \frac{\beta}{\alpha}\rho_X(x_{i-1},x_i) \leq \frac{\beta}{\alpha}q_0\rho_X(x_i,x_{i-1}).$$

Thus,

$$\rho_X(x_{i+1},x_i) \leq q_0^2\frac{\beta}{\alpha}\rho_X(x_i,x_{i-1}). \tag{2.11}$$

Let

$$\tilde{a} = \rho_X(x_1,x_0), \quad \tilde{\beta} = q_0^2\frac{\beta}{\alpha}.$$

Then, we have $\tilde{\beta} < 1$. By inequality (2.11), we have:

$$\begin{aligned}\rho_X(x_{i+1},x_i) \leq \tilde{\beta}\rho_X(x_i,x_{i-1}) \leq \tilde{\beta}^2\rho_X(x_{i-1},x_{i-2}) \leq \cdots \\ \leq \tilde{\beta}^i d_X(x_1,x_0) \leq \tilde{\beta}^i\tilde{a}.\end{aligned} \tag{2.12}$$

For all integers $i, j \in \mathbb{N}$, using (2.11), (2.12), we obtain:

$$\begin{aligned}\rho_X(x_{i+j},x_i) &\leq q_1\rho_X(x_{i+j},x_{i+1}) + q_2\rho_X(x_{i+1},x_i) \\ &\leq q_1\rho_X(x_{i+j},x_{i+1}) + q_2\tilde{\beta}^i\tilde{a} \\ &\leq q_1\big(q_1\rho_X(x_{i+j},x_{i+2}) + q_2\rho_X(x_{i+2},x_{i+1})\big) + q_2\tilde{\beta}^i\tilde{a} \\ &\leq q_1^2\rho_X(x_{i+j},x_{i+2}) + q_1q_2\rho_X(x_{i+2},x_{i+1}) + q_2\tilde{\beta}^i\tilde{a} \\ &\leq q_1^2\big(q_1\rho_X(x_{i+j},x_{i+3}) + q_2\rho_X(x_{i+3},x_{i+2})\big) + q_1q_2\tilde{\beta}^{i+1}\tilde{a} + q_2\tilde{\beta}^i\tilde{a} \\ &\leq q_1^3\rho_X(x_{i+j},x_{i+3}) + q_1^2q_2d_X(x_{i+3},x_{i+2}) + q_1q_2\tilde{\beta}^{i+1}\tilde{a} + q_2\tilde{\beta}^i\tilde{a}.\end{aligned}$$

Having carried out similar transformations a sufficient number of times, we obtain

$$\begin{aligned}\rho_X(x_{i+j},x_i) &\leq q_2\tilde{\beta}^i\tilde{a} + q_1q_2\tilde{\beta}^{i+1}\tilde{a} + q_1^2q_2\tilde{\beta}^{i+2}\tilde{a} + \cdots + q_1^{j-2}q_2\tilde{\beta}^{i+(j-2)}\tilde{a} + q_1^{j-1}\tilde{\beta}^{i+(j-1)}\tilde{a} \\ &\leq q_2\tilde{\beta}^i\tilde{a}(1 + q_1\tilde{\beta}^1 + q_1^2\tilde{\beta}^2 + \cdots + q_1^{j-2}\tilde{\beta}^{j-2} + q_1^{j-1}\tilde{\beta}^{j-1}q_2^{-1}) = q_2\tilde{\beta}^i\tilde{a}\hat{S}(j),\end{aligned}$$

where

$$\hat{S}(j) = S(q_1\tilde{\beta}, j-1) + q_1^{j-1}\tilde{\beta}^{j-1}q_2^{-1}, \quad j \in \mathbb{N}, \quad \hat{S}(0) = 0.$$

From the latter, for all non-negative integers i and k, we obtain:

$$\begin{aligned}\rho_X(x_{i+k},x_i) &\le q_1\rho_X(x_{i+k},x_{i+n_0})+q_2\rho_X(x_{i+n_0},x_i)\\ &\le q_1\big(q_1 d_X(x_{i+k},x_{i+2n_0})+q_2\rho_X(x_{i+2n_0},x_{i+n_0})\big)+q_2\rho_X(x_{i+n_0},x_i)\\ &\le q_1^2\rho_X(x_{i+k},x_{i+2n_0})+q_1q_2\rho_X(x_{i+2n_0},x_{i+n_0})+q_2\rho_X(x_{i+n_0},x_i)\\ &\le q_1^2\big(q_1\rho_X(x_{i+k},x_{i+3n_0})+q_2\rho_X(x_{i+3n_0},x_{i+2n_0})\big)+q_1q_2\rho_X(x_{i+2n_0},x_{i+n_0})\\ &\qquad +q_2\rho_X(x_{i+n_0},x_i)\\ &\le q_1^3\rho_X(x_{i+k},x_{i+3n_0})+q_1^2q_2\rho_X(x_{i+3n_0},x_{i+2n_0})+q_1q_2\rho_X(x_{i+2n_0},x_{i+n_0})\\ &\qquad +q_2\rho_X(x_{i+n_0},x_i).\end{aligned}$$

We continue similar calculations, and as the result we get

$$\begin{aligned}\rho_X(x_{i+k},x_i) &\le q_2^2\tilde\beta^i\tilde a\hat S(n_0)+q_1q_2^2\tilde\beta^{i+n_0}\tilde a\hat S(n_0)+q_1q_2^2\tilde\beta^{i+2n_0}\tilde a\hat S(n_0)+\cdots\\ &\qquad +q_1^{r-1}q_2^2\tilde\beta^{i+(r-1)n_0}\tilde a\hat S(n_0)+q_1^r\rho_X(x_{i+k},x_{i+rn_0})\\ &\le q_2^2\tilde\beta^i\tilde a\hat S(n_0)\big(1+q_1\tilde\beta^{n_0}+q_1\tilde\beta^{2n_0}+\cdots+q_1^{r-1}\tilde\beta^{(r-1)n_0}\big)+q_1^r\rho_X(x_{i+k},x_{i+rn_0})\\ &\le q_1^rq_2\tilde\beta^{i+rn_0}\tilde a\hat S(k-rn_0)+q_2^2\tilde\beta^i\tilde a\hat S(n_0)S(q_1\tilde\beta^{n_0},r),\end{aligned}$$

where $r=[k/n_0]$.

Thus, considering that by construction

$$q_1\tilde\beta^{n_0}<1,$$

we have:

$$\begin{aligned}\rho_X(x_{i+k},x_i) &\le q_2^2\tilde\beta^i\tilde a\big(q_1^rq_2^{-1}\tilde\beta^{rn_0}\hat S(k-rn_0)+\hat S(n_0)S(q_1\tilde\beta^{n_0},r)\big)\\ &\le q_2^2\tilde\beta^i\tilde a\left(q_2^{-1}\hat S(k-rn_0)+\frac{\hat S(n_0)}{1-q_1\tilde\beta^{n_0}}\right).\end{aligned}\tag{2.13}$$

If $k\in\mathbb{N}$ is a multiple of n_0, then by construction,

$$\hat S(k-rn_0)=0.$$

Thus, for $k\in\mathbb{N}$, a multiple of n_0 from (2.13), for $i=0$, we have

$$\begin{aligned}\rho_X(x_k,x_0) &\le \frac{q_2^2\hat S(n_0)}{1-q_1\tilde\beta^{n_0}}\rho_X(x_1,x_0)\\ &\le \frac{1}{\alpha}\frac{q_0q_2^2\hat S(n_0)}{1-q_1\tilde\beta^{n_0}}d(\psi(x_0),\varphi(x_0)),\end{aligned}\tag{2.14}$$

since

$$\hat a\le\frac{q_0}{\alpha}\,d(\psi(x_0),\varphi(x_0)).$$

Now, using (2.6), and from inequality (2.14) as in the previous section, using (1.5) we obtain

$$\rho_X(\xi,x_0)\le\frac{q_0}{\alpha}\frac{q_2^3\hat S(n_0)}{1-q_1\tilde\beta^{n_0}}d(\psi(x_0),\varphi(x_0)).$$

This proves (2.3).

From (2.3) also, by using (2.14), we obtain:

$$\varliminf_{\eta \to \xi} \rho_X(\eta, x_0) \leq \frac{q_0}{\alpha} \frac{q_2^2 \hat{S}(n_0)}{1 - q_1 \tilde{\beta}^{n_0}} d(\psi(x_0), \varphi(x_0)).$$

This proves (2.4). □

We now study the stability of the coincidence point ξ of the mappings ψ, φ. We will interpret stability as follows. Let $\psi_n, \varphi_n : X \to Y$ be defined for all $n \in \mathbb{N}$. Assuming some convergence of these mappings to the original mappings ψ, φ, we will show that for any n, the mappings ψ_n, φ_n have a coincidence point ξ_n, and ξ_n converges to ξ.

Recall that a coincidence point of the mappings $\psi_n, \varphi_n : X \to Y$ is the solution of the equation

$$\psi_n(x) = \varphi_n(x).$$

Theorem 2.2. *Let (X, ρ_X) be a complete weakly symmetric (q_1, q_2)-quasimetric space, and let $\xi \in X$ be a coincidence point of the mappings $\psi, \varphi : X \to Y$. For each $n \in \mathbb{N}$, given $0 \leq \beta_n < \alpha_n$, we define:*

$$S_n = \frac{q_1^3 \alpha_n^{m_0 - 1} S(q_2 \frac{\beta_n}{\alpha_n}, m_0 - 1) + q_1^2 (q_2 \beta_n)^{m_0 - 1}}{\alpha_n^{m_0} - q_2 \beta_n^{m_0}} d(\psi_n(\xi), \varphi_n(\xi)).$$

We assume that for each $n \in \mathbb{N}$, $x \in X$, the mapping ψ_n is α_n-covering and closed, the mapping φ_n is β_n-Lipschitz. If $S_n \to 0$ as $n \to \infty$, then for all $n \in \mathbb{N}$, equation (2.3) has at least one solution ξ_n such that $\xi_n \to \xi$ as $n \to \infty$.

P r o o f. For every $n \in \mathbb{N}$, the conditions of Theorem 2.1 are satisfied for the mappings $\psi_n, \varphi_n : X \to Y$, with $x_0 = \xi$. So for each n, there exists a solution ξ_n for the equation $\psi_n(x) = \varphi_n(x)$ such that $\rho_X(\xi, \xi_n) \leq S_n$. Due to the convergence $S_n \to 0$ as $n \to \infty$, we obtain $\xi_n \to \xi$ as $n \to \infty$. □

We note that this theorem is an extension of the result obtained in [8].

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