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ON POSITIVITY OF THE GREEN FUNCTION FOR POISSON PROBLEM FOR A LINEAR FUNCTIONAL DIFFERENTIAL EQUATION

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For the Poisson problem

$$-\Delta u + p(x)u - \int_\Omega u(s)\, r(x, ds) = \rho f, \quad u\big|_{\Gamma(\Omega)} = 0$$

equivalence of positivity of the Green function and other classical properties is showed. Here Ω is an open set in $\mathbb{R}^n$, and $\Gamma(\Omega)$ is the boundary of the Ω. For almost all $x \in \Omega$, $r(x, \cdot)$ is a measure satisfying certain symmetry condition. In particular this equation involves integral differential equation and the equation

$$-\Delta u + p(x)u(x) - \sum_{i=1}^{m} p_i(x)u(h_i(x)) = \rho f,$$

where $h_i \colon \Omega \to \Omega$ is a measurable mapping.
Keywords: Green function; Poisson problem; Vallee-Poussin theorem; Spectrum of selfadjoint operator

1. The Poisson problem

1.1. The problem

Let Ω be an open set in $\mathbb{R}^n$, and $\Gamma(\Omega)$ be the boundary of the Ω. For a function $u = u(x)$, $x \in \Omega$, let $\Delta u = u''_{x_1x_1} + \dots + u''_{x_nx_n}$, where $x = (x_1, \dots, x_n)$. In the Poisson problem

$$-\Delta u + p(x)u - \int_\Omega u(s)\, r(x, ds) = \rho(x)f(x), \tag{1}$$

$$u\big|_{\Gamma(\Omega)} = 0 \tag{2}$$

for almost all $x \in \Omega$, the function $r(x, \cdot)$ is a measure satisfying certain symmetry condition. For example, if for each x the measure is concentrated at the points $h_i(x)$, $i = 1, \dots, m$, the equation (1) will have the form

$$-\Delta u + p(x)u(x) - \sum_{i=1}^{m} p_i(x)u(h_i(x)) = \rho(x)f(x),$$

where $h_i \colon \Omega \to \Omega$ is a measurable mapping. The function ρ is a positive weight.

The functional differential equation (1) has certain mechanical interpretation if $p(x) \geq r(x, \Omega)$. For the case $n=2$ it describes the state of the loaded membrane with added special internal forces. This fact allows predict positivity of the Green function [1]. Here the condition $p(x) \geq r(x, \Omega)$ is omitted. The boundary value problem (BVP) (1), (2) has the Fredholm property. In case of unique solvability its solution can be represented by means of the Green's function

$$u(x) = \int_\Omega G(x,s) f(s)\, ds.$$

1.2. Assumptions and notation

Let $\Omega \subset \mathbb{R}^n$ be a nonempty *bounded open* set, $\Gamma(\Omega)$ be the boundary of the Ω, and $X = \overline{\Omega}$ be the closure of Ω. For a real function $u = u(x)$ defined on Ω and having derivative of first order,[1] $u'_x := \left(u'_{x_1}, \ldots, u'_{x_n}\right)$, where $x = (x_1, \ldots, x_n)$. For two such functions u and v,

$$u'_x v'_x := u'_{x_1} v'_{x_1} + \cdots + u'_{x_n} v'_{x_n}.$$

Let's consider the following two bilinear forms[2]

$$[u,v] := \int\limits_\Omega u'_x v'_x\, dx + \int\limits_\Omega p(x) u(x) v(x)\, dx - \int\limits_{\Omega\times\Omega} v(x) u(s)\, \xi_0(dx \times ds), \tag{3}$$

($dx := dx_1 \cdots dx_n$) and

$$\langle u, v\rangle := [u,v] - \int\limits_{\Omega\times\Omega} v(x) u(s) \eta(dx \times ds). \tag{4}$$

The domain Ω is assumed to satisfy the *cone condition* [2].

1.2.1. The form (3) we use under following assumptions

Let $\mathcal{M}$ be the set of all Lebesgue measurable subsets in $X = \overline{\Omega}$. Let the function $r\colon X \times \mathcal{M} \to$ $\to R$ satisfy two conditions: for almost all $x \in X$, the function $r_0(x, \cdot)$ is a measure on $\mathcal{M}$, for any $e \in \mathcal{M}$, $r_0(\cdot, e)$ is measurable on X. Let

$$p(x) := r_0(x, \Omega).$$

The set function ξ_0 defined by the equality

$$\xi_0(E) = \int_X r_0(x, E_x)\, dx,\ E_x = \{y\colon (x,y) \in E\}$$

is a measure. Assume that ξ_0 is symmetric, that is,

$$\xi_0(e_1 \times e_2) = \xi_0(e_2 \times e_1),\ \forall e_1, e_2 \in \mathcal{M}.$$

The measure η has the same properties and is defined by

$$\eta(E) = \int_X q(x, E_x)\, dx,\ E_x = \{y\colon (x,y) \in E\},$$

where q has properties analogous to r_0. The measures ξ and $r(x, \cdot)$ define by

$$\xi := \xi_0 + \eta,\ r := r_0 + q$$

[1] $:=$ signifies 'is equal by definition'

[2] the notation $\xi(dx \times dy)$ is equivalent to $d\xi$

1.2.2. Two main spaces

We use the Sobolev spaces $W_0^{1,2}(\Omega)$ and $W_0^{2,2}(\Omega)$ [2].

D e f i n i t i o n 1.1. Let W be the vector subspace of all elements from $W_0^{1,2}(\Omega)$ satisfying $[u,u]<\infty$.

The bilinear form $[u,v]$ is an inner product in the Hilbert space W.

Let $\rho(x)$, $x\in X$, be a positive measurable and integrable in Ω function and $\mu(E):=\int_E \rho(x)\,dx$. Let

$$(f,g)=\int\limits_{\Omega} f(x)g(x)\rho(x)\,dx$$

and $L_2(X,\mu)$ (or $L_2(\Omega,\mu)$) be the Hilbert space of all μ-measurable functions on X with finite integral $\int\limits_{\Omega} f(x)^2\rho(x)\,dx$.

Define the operator $T\colon W\to L_2(\Omega,\mu)$ by the equality $Tu(x)=u(x)$, $x\in\Omega$. The operator T is continuous.

1.3. A variational form of the problem. Euler equation

The equation with relation to u in variational form

$$\int\limits_{\Omega} u'_x v'_x\,dx+\int\limits_{\Omega} p(x)u(x)v(x)\,dx-\int\limits_{\Omega\times\Omega} v(x)u(s)\,\xi_0(dx\times ds),$$

$$=\int_{\Omega} f(x)v(x)\rho(x)\,dx,\quad \forall v\in W,$$

can be represented in the short form

$$[u,v]=(f,Tv),\quad (\forall v\in W).\tag{5}$$

The image $T(W)$ is dense in $L_2(X,\mu)$. The operator T^* has an inverse $\mathcal{L}_0$ [3]. For any $f\in L_2(\Omega,\mu)$ the equation (5) has a unique solution $u=T^*f\in W$.

The set $T^*(L_2(X,\mu))$ is the domain of the operator $\mathcal{L}_0$.

For any $u\in C_0^\infty(\Omega)$ or $u\in W_0^{2,2}(\Omega)$ and $v\in W$ integrating by parts obtain the identity

$$\int_{\Omega} u'_x v'_x\,dx=-\int_{\Omega}\Delta u\cdot v\,dx.$$

Hence, if $\Delta u=g$, then

$$\int_{\Omega} u'_x v'_x\,dx=-\int_{\Omega} g\cdot v\,dx,\quad \forall v\in W.\tag{6}$$

Since $C_0^\infty(\Omega)\subset D(\mathcal{L}_0)$ the equation (6) can be used as *definition of operator* Δ on the space $D(\mathcal{L}_0)$ in a weak sense.

P r o p o s i t i o n 1.1. Operator $\mathcal{L}_0$ has representation

$$\mathcal{L}_0 u=\frac{1}{\rho}\left(-\Delta u+p(x)u-\int_{\Omega} u(s)\,r_0(x,ds)\right).\tag{7}$$

[3] operator $\mathcal{L}_0$ can be considered as extension of the operator defined by (7)

2. Results

2.1. Eigenvalue problem and spectrum

Theorem 2.1. The eigenvalue problem

$$-\Delta u + pu - \int\limits_{\Omega} u(s)\, r_0(x,ds) = \lambda \rho u, \quad u\big|_{\Gamma(\Omega)} = 0 \tag{8}$$

has in W a system of nontrivial solutions $u_n(x)$ corresponding to positive eigenvalues λ_n. That is, $\lambda_0 \le \lambda_1, \ldots$. This system forms an orthogonal basis in the space $W_0^{1,2}$.

Note that the minimal eigenvalue λ_0 satisfies the relation

$$\lambda_0 = \inf_{[u,u]=1} \frac{[u,u]}{(Tu,Tu)}.$$

2.2. Positivity of solutions

The problem

$$-\Delta u + pu - \int\limits_{\Omega} u(s)\, r_0(x,ds) = \rho f, \quad u\big|_{\Gamma(\Omega)} = 0 \tag{9}$$

represents the equation $\mathcal{L}_0 u = f$.

Theorem 2.2. Suppose $f \geq\not\equiv 0$ and $u(x)$ is the solution of the problem (9). Then $u(x) > 0$ in Ω.

Corollary 2.1. The minimal eigenvalue λ_0 of the problem (8) is positive and simple $(\lambda_0 < \lambda_1)$. It associated with a positive in Ω eigenfunction $u_0(x)$.

2.3. General case

Here we consider the form (4). The equation in variational form

$$\langle u, v\rangle = (f, Tv), \quad \forall v \in W,$$

is equivalent to the boundary value problem

$$\mathcal{L}u := \mathcal{L}_0 u - Qu = f, \; u\big|_{\Gamma(\Omega)} = 0, \tag{10}$$

where the operator $Q\colon W \to L_2(\Omega,\mu)$ has the representation

$$Qu(x) = (1/\rho) \int_{\Omega} u(s) q(x,ds).$$

We may to impose some conditions to ensure the action $Q\colon W \to L_2(\Omega,\mu)$ and continuity. In this case the operator QT^* will be compact. It may be showed that this operator will be compact if

$$q(\cdot,\Omega) \in L_2(\Omega,\rho). \tag{11}$$

Theorem 2.3. Suppose (11) is fulfilled. The eigenvalue problem

$$-\Delta u + pu - \int\limits_{\Omega} u(s)\, r(x,ds) = \lambda \rho u, \quad u\big|_{\Gamma(\Omega)} = 0 \tag{12}$$

has in W a system of nontrivial solutions $u_n(x)$ corresponding to eigenvalues $\lambda_0 \le \lambda_1 \le \ldots$. This system forms an orthogonal basis in the spaces $W_0^{1,2}$ and in W, and in $L_2(\Omega,\rho)$.

Theorem 2.4. The following affirmations are equivalent:

1. the quadratic functional $\langle u, u\rangle$ defined by (4) is positive definite,
2. the problem (10) is uniquely resolvable, and its Green function is positive in $\Omega \times \Omega$,
3. the inequality $-\Delta v + pv - \int_\Omega v(s)\, r(x, ds) \geq\not\equiv 0$ has positive in Ω solution,
4. the minimal eigenvalue of the problem (12) is positive,
5. spectral radius of the operator QT^* is less than unit.

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О ПОЛОЖИТЕЛЬНОСТИ ФУНКЦИИ ГРИНА ДЛЯ ЗАДАЧИ ПУАССОНА ДЛЯ ЛИНЕЙНОГО ФУНКЦИОНАЛЬНО-ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ

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Для задачи Пуассона

$$-\Delta u + p(x)u - \int_\Omega u(s)\, r(x, ds) = \rho f, \quad u\big|_{\Gamma(\Omega)} = 0$$

показана эквивалентность положительности функции Грина и других классических свойств. Здесь Ω – открытое множество в $\mathbb{R}^n$, и $\Gamma(\Omega)$ – граница Ω. Для почти всех $x \in \Omega$, $r(x, \cdot)$ – мера, удовлетворяющая некоторому условию симметрии. В частности, это уравнение охватывает интегро-дифференциальное уравнение и уравнение

$$-\Delta u + p(x)u(x) - \sum_{i=1}^{m} p_i(x)u(h_i(x)) = \rho f,$$

где $h_i : \Omega \to \Omega$ – измеримое отображение.

Ключевые слова: функция Грина; задача Пуассона; теорема Валле-Пуссена; спектр самосопряженного оператора

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