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# BEREZIN QUANTIZATION AS A PART OF THE REPRESENTATION THEORY

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We present an approach to polynomial quantization (a variant of quantization in the spirit of Berezin) on para-Hermitian symmetric spaces using the notion of an "overgroup". This approach gives covariant and contravariant symbols and the Berezin transform in a highly natural and transparent way.
*Keywords:* symplectic manifolds; symbol calculus; quantization; Berezin transform

Let $G/H$ be a para-Hermitian symmetric space. We can consider that $G/H$ is a manifold in the Lie algebra $\mathfrak{g}$ of $G$. Quantization on $G/H$ in the spirit of Berezin has been constructed in [6]. Polynomial quantization (a variant of quantization) has been given in [7], with explicit formulae for rank one spaces. Here an initial algebra operators is the algebra of operators in a maximal degenerate series representation of the universal enveloping algebra of $\mathfrak{g}$. In this paper we suggest a new approach to polynomial quantization using the notion of an "overgroup", see § 5. Here covariant and contravariant symbols and the Berezin transform appear quite naturally and transparently. Moreover, such a point of view can bring to different generalizations of the theory.

## § 1. Para-Hermitian symmetric spaces

Let $G/H$ be a *semisimple symmetric space*. Here $G$ is a connected semisimple Lie group with an involutive automorphism $\sigma \neq 1$, and $H$ is an open subgroup of $G^\sigma$, the subgroup of fixed points of $\sigma$. We consider that groups act on their homogeneous spaces *from the right*, so that $G/H$ consists of right cosets $Hg$.

Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of $G$ and of $H$ respectively. Let $B_\mathfrak{g}$ be the Killing form of $G$. There is a decomposition of $\mathfrak{g}$ into direct sums of $+1$, $-1$-eigenspaces of the involution $\sigma$:

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{q}.$$

The subspace $\mathfrak{q}$ is invariant with respect to $H$ in the adjoint representation $\mathrm{Ad}$. It can be identified with the tangent space to $G/H$ at the point $x^0 = He$.

The dimension of Cartan subspaces of $\mathfrak{q}$ (maximal Abelian subalgebras in $\mathfrak{q}$ consisting of semisimple elements) is called the rank of $G/H$.

Now let $G/H$ be a *symplectic* manifold. Then $\mathfrak{h}$ has a non-trivial centre $Z(\mathfrak{h})$. For simplicity we assume that $G/H$ is an orbit $\mathrm{Ad}\, G \cdot Z_0$ of an element $Z_0 \in \mathfrak{g}$. In particular, then $Z_0 \in Z(\mathfrak{h})$.

Further, we can also assume that $G$ is *simple*. Such spaces $G/H$ are divided into 4 classes (see [3], [4]):

$(a)$ Hermitian symmetric spaces;
$(b)$ semi-Kählerian symmetric spaces;
$(c)$ para-Hermitian symmetric spaces;
$(d)$ complexifications of spaces of class (a).

Dimensions of $Z(\mathfrak{h})$ are 1,1,1,2, respectively. Spaces of class $(a)$ are Riemannian, of other three classes are pseudo-Riemannian (not Riemannian).

We focus on spaces of class $(c)$. Here the center $Z(\mathfrak{h})$ is one-dimensional, so that $Z(\mathfrak{h}) = \mathbb{R}Z_0$, and $Z_0$ can be normalized so that the operator $I = (\mathrm{adZ}_0)_\mathfrak{q}$ on $\mathfrak{q}$ has eigenvalues $\pm 1$. A symplectic structure on $G/H$ is defined by the bilinear form $\omega(X,Y) = B_\mathfrak{g}(X, IY)$ on $\mathfrak{q}$.

The $\pm 1$-eigenspaces $\mathfrak{q}^{\pm} \subset \mathfrak{q}$ of $I$ are Lagrangian, $H$-invariant, and irreducible. They are Abelian subalgebras of $\mathfrak{g}$. So $\mathfrak{g}$ becomes a graded Lie algebra:

$$\mathfrak{g} = \mathfrak{q}^- + \mathfrak{h} + \mathfrak{q}^+,$$

with commutation relations $[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}$, $[\mathfrak{h},\mathfrak{q}^-] \subset \mathfrak{q}^-$, $[\mathfrak{h},\mathfrak{q}^+] \subset \mathfrak{q}^+$.

The pair $(\mathfrak{q}^+, \mathfrak{q}^-)$ is a Jordan pair with multiplication

$$\{XYZ\} = \frac{1}{2}\,[[X,Y],\,Z],$$

see [5]. Let $r$ and $\varkappa$ be the rank and the genus of this Jordan pair. The rank $r$ coincides with the rank of $G/H$.

Set $Q^{\pm} = \exp \mathfrak{q}^{\pm}$. The subgroups $P^{\pm} = HQ^{\pm} = Q^{\pm}H$ are maximal parabolic subgroups of $G$. One has the following decompositions:

$$G \ = \overline{Q^+HQ^-} \tag{1.1}$$

$$= \overline{Q^-HQ^+}, \tag{1.2}$$

where bar means closure and the sets under the bar are open and dense in $G$. Let us call (1.1) and (1.2) the *Gauss decomposition* and (allowing some slang) the *anti-Gauss decomposition* respectively. Decompositions (1.1), (1.2) mean that almost any element $g \in G$ can be decomposed as ( the Gauss decomposition):

$$g = \exp\eta \cdot h \cdot \exp\xi, \tag{1.3}$$

or (the anti-Gauss decomposition):

$$g = \exp\xi \cdot h \cdot \exp\eta, \tag{1.4}$$

where $h \in H$, $\xi \in \mathfrak{q}^-$, $\eta \in \mathfrak{q}^+$, all three factors in (1.3) and (1.4) are defined uniquely. We also use the Gauss decomposition (1.3) in a little different form:

$$g = \exp\eta \cdot \exp\xi \cdot h, \tag{1.5}$$

where $\eta$ and $h$ are the same as in (1.3), and $\xi$ is obtained from $\xi$ in (1.3) by $\mathrm{Ad}\, h$.

Decompositions (1.3) and (1.4) generate actions of $G$ on $\mathfrak{q}^-$ and $\mathfrak{q}^+$ respectively, namely, $\xi \mapsto \widetilde{\xi} = \xi \bullet g$ and $\eta \mapsto \widehat{\eta} = \eta \circ g$:

$$\exp\xi \cdot g = \exp Y \cdot \widetilde{h} \cdot \exp\widetilde{\xi}, \tag{1.6}$$

$$\exp\eta \cdot g = \exp X \cdot \widehat{h} \cdot \exp\widehat{\eta}, \tag{1.7}$$

where $X \in \mathfrak{q}^-$, $Y \in \mathfrak{q}^+$. These actions are defined on open and dense sets depending on $g$. Therefore, $G$ acts on $\mathfrak{q}^- \times \mathfrak{q}^+$: $(\xi,\eta) \mapsto (\widetilde{\xi}, \widehat{\eta})$. The stabilizer of the point $(0,0) \in \mathfrak{q}^- \times \mathfrak{q}^+$ is $P^+ \cap P^- = H$, so that we get an embedding

$$\mathfrak{q}^- \times \mathfrak{q}^+ \hookrightarrow G/H. \tag{1.8}$$

It is defined on an open and dense set, its image is also an open and dense set. Therefore, we can consider $(\xi,\eta)\in\mathfrak{q}^-\times\mathfrak{q}^+$ as coordinates on $G/H$, let us call them *horospherical coordinates.*

Let us write explicit formula for embedding (1.8). We use a redecomposition "anti-Gauss" to "Gauss". We take $\xi\in\mathfrak{q}^-$, $\eta\in\mathfrak{q}^+$ and decompose the anti-Gauss product $\exp\xi\cdot\exp(-\eta)$ according to formula (1.5) (the "Gauss"):

$$\exp\xi\cdot\exp(-\eta)=\exp Y\cdot\exp X\cdot h, \tag{1.9}$$

where $X\in\mathfrak{q}^-$, $Y\in\mathfrak{q}^+$. The obtained element $h\in H$ depends on $\xi$ and $\eta$ only, denote it by $h(\xi,\eta)$. Using (1.9), let us form the following element $g\in G$:

$$g=\exp Y\exp\xi=\exp X\cdot h\cdot\exp\eta. \tag{1.10}$$

Then the pair $\xi,\eta$ goes just to the point $x=x^0g$ where $g$ is defined by (1.10).

Under the action of the group $G$ the element $h(\xi,\eta)$ is transformed as follows:

$$h(\widetilde{\xi},\widehat{\eta})=\widetilde{h}^{-1}\cdot h(\xi,\eta)\cdot\widehat{h}, \tag{1.11}$$

where $\widetilde{h}$ and $\widehat{h}$ are taken from (1.6) and (1.7) respectively.

For $h\in H$, denote

$$b(h)=\det(\mathrm{Ad}\,h)|_{\mathfrak{q}^+}.$$

The function $k(\xi,\eta)=b(h(\xi,\eta))$ is $N(\xi,\eta)^{-\varkappa}$, where $N(\xi,\eta)$ is an irreducible polynomial $N(\xi,\eta)$ of degree $r$ in $\xi$ and in $\eta$ separately. Considered as a function on $G/H$, the function $k(\xi,\eta)$ is an analogue of the Bergman kernel for Hermitian symmetric spaces. It follows from (1.11) that under action of $g\in G$ the function $N(\xi,\eta)$ is transformed as follows:

$$N(\widetilde{\xi},\widehat{\eta})=N(\xi,\eta)\cdot b(\widetilde{h})^{1/\varkappa}\cdot b(\widehat{h})^{-1/\varkappa} \tag{1.12}$$

In horospherical coordinates the $G$-invariant measure on $G/H$ is:

$$dx=dx(\xi,\eta)=|N(\xi,\eta)|^{-\varkappa}\,d\xi\,d\eta,$$

where $d\xi$ and $d\eta$ are Euclidean measures on $\mathfrak{q}^-$ and $\mathfrak{q}^+$ respectively.

## § 2. Maximal degenerate series representations

In this Section we introduce two series of representations induced by characters of maximal parabolic subgroups $P^\pm$ of $G$ (maximal degenerate series representations).

First we take the character $\omega_\lambda(h)$, $\lambda\in\mathbb{C}$, of $H$:

$$\omega_\lambda(h)=\left|b(h)\right|^{-\lambda/\varkappa}$$

and then we extend this character to the subgroups $P^\pm$, setting it equal to 1 on $Q^\pm$.

Let us consider induced representations $\pi_\lambda^\pm$ of $G$:

$$\pi_\lambda^\pm=\mathrm{Ind}\left(G,\,P^\mp,\,\omega_{\mp\lambda}\right).$$

They act on the space $\mathcal{D}_\lambda^\pm(G)$ of functions $f\in C^\infty(G)$ having the uniformity property:

$$f(pg)=\omega_{\mp\lambda}(p)f(g),\quad p\in P^\mp,$$

by translations from the right:

$$\left(\pi_\lambda^\pm(g)f\right)(g_1) = f(g_1 g).$$

Realize them in the *non-compact picture*: we restrict functions from $\mathcal{D}_\lambda^\pm(G)$ to the subgroups $Q^\pm$ and identify them (as manifolds) with $\mathfrak{q}^\pm$, we obtain

$$\left(\pi_\lambda^-(g)f\right)(\xi) = \omega_\lambda(\widetilde{h})f(\widetilde{\xi}), \quad \left(\pi_\lambda^+(g)f\right)(\eta) = \omega_\lambda(\widehat{h}^{-1})f(\widehat{\eta}),$$

where $\widetilde{\xi}$, $\widetilde{h}$, $\widehat{\eta}$, $\widehat{h}$ are taken from decompositions (1.6), (1.7).

Let us write intertwining operators. Introduce operators $A_\lambda$ and $B_\lambda$ by:

$$\begin{aligned}(A_\lambda\varphi)(\eta) &= \int_{\mathfrak{q}^-} |N(\xi,\eta)|^{-\lambda-\varkappa}\, \varphi(\xi)\, d\xi,\\ (B_\lambda\varphi)(\xi) &= \int_{\mathfrak{q}^+} |N(\xi,\eta)|^{-\lambda-\varkappa}\, \varphi(\eta)\, d\eta.\end{aligned}$$

The operator $A_\lambda$ intertwines $\pi_\lambda^-$ with $\pi_{-\lambda-\varkappa}^+$ and the operator $B_\lambda$ intertwines $\pi_\lambda^+$ with $\pi_{-\lambda-\varkappa}^-$. Their composition is a scalar operator:

$$B_\lambda A_{-\lambda-\varkappa} = c(\lambda)^{-1}\cdot \mathrm{id}, \tag{2.1}$$

where $c(\lambda)$ is a meromorphic function of $\lambda$, it is invariant with respect to the change $\lambda \mapsto -\lambda-\varkappa$.

We can extend $\pi_\lambda^\pm$, $A_\lambda$ and $B_\lambda$ to distributions on $\mathfrak{q}^-$ and $\mathfrak{q}^+$.

The representation $\pi_\lambda^-$ of the Lie algebra $\mathfrak{g}$ is given by some differential operators of the first order. This representation can be considered on different spaces of functions of $\xi$: for example, the space $C^\infty(\mathfrak{q}^-)$, the space $\mathrm{Pol}\,(\mathfrak{q}^-)$ of polynomials in $\xi$, the space $\mathcal{D}'(\mathfrak{q}^-)$ of distributions on $\mathfrak{q}^-$, in particular, the space $Z$ of distributions on $\mathfrak{q}^-$ concentrated at the origin, etc. The same concerns to $\pi_\lambda^+$.

Notice

$$\omega_\lambda\big(h(\xi,\eta)\big) = |N(\xi,\eta)|^\lambda, \tag{2.2}$$

hence formula (1.12) gives

$$|N(\widetilde{\xi},\widehat{\eta})|^\lambda = |N(\xi,\eta)|^\lambda\cdot\omega_\lambda(\widetilde{h})^{-1}\cdot\omega_\lambda(\widehat{h}),$$

which can be interpreted as an invariance property of the function $|N(\xi,\eta)|^\lambda$ :

$$\Big[\pi_\lambda^-(g)\otimes\pi_\lambda^+(g)\Big]|N(\xi,\eta)|^\lambda = |N(\xi,\eta)|^\lambda.$$

## § 3. Symbols and transforms

In this Section we apply to a para-Hermitian symmetric space $G/H$ the scheme of quantization in the spirit of Berezin offered in [6]. We consider a variant of the quantization, we call the *polynomial quantization*. For an initial algebra of operators we take here the algebra of operators $\pi_\lambda^-(\mathrm{Env}\,(\mathfrak{g}))$, where $\lambda\in\mathbb{C}$ and $\mathrm{Env}\,(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$. In contrast to [6], we use the non-compact picture for representations $\pi_\lambda^\pm$, see § 2. The role of the Fock space is played by a space of functions $\varphi(\xi)$, $\xi\in\mathfrak{q}^-$, so that our operators act in functions $\varphi(\xi)$. We introduce covariant and contravariant symbols of operators, the Berezin transform etc.

As a (an analogue of) *supercomplete system* we take the kernel of the intertwining operators $A_{-\lambda-\varkappa}$ from § 2, namely, the function

$$\Phi(\xi,\eta) = \Phi_\lambda(\xi,\eta) = |N(\xi,\eta)|^\lambda . \tag{3.1}$$

For an operator $D = \pi_\lambda^-(X)$, $X \in \mathrm{Env}\,(\mathfrak{g}))$, the function

$$F(\xi,\eta) = \frac{1}{\Phi(\xi,\eta)}\,(\pi_\lambda^-(X)\otimes 1)\Phi(\xi,\eta)$$

is called the *covariant symbol* of $D$. Since $\xi$, $\eta$ are horospherical coordinates on $G/H$, covariant symbols become functions on $G/H$ and, moreover, *polynomials* on $G/H \subset \mathfrak{g}$. It is why we call this variant of quantization the *polynomial* quantization. Denote the space of covariant symbols by $\mathcal{A}_\lambda$. For generic $\lambda$, this space is the space $S(G/H)$ of *all polynomials* on $G/H$.

In particular, the covariant symbol of the identity operator is the function on $G/H$ equal to $1$ identically. If $X$ belongs to the Lie algebra $\mathfrak{g}$ itself, then the covariant symbol of the operator $\pi_\lambda^-(X)$ is a linear function $B_\mathfrak{g}(X,x)$ of $x \in G/H \subset \mathfrak{g}$ with coordinates $\xi$, $\eta$, up to a factor depending on $\lambda$.

The operator $D$ is recovered by its covariant symbol $F$:

$$(D\varphi)(\xi) = c\int_{G/H} F(\xi,v)\,\frac{\Phi(\xi,v)}{\Phi(u,v)}\,\varphi(u)\,dx(u,v), \tag{3.2}$$

where $c = c(\lambda)$ is taken from (2.1). Indeed, the function $\Phi$ has a reproducing property

$$\varphi(\xi) = c(\lambda)\int_{G/H} \frac{\Phi(\xi,\eta)}{\Phi(u,v)}\,\varphi(u)\,dx(u,v).$$

which is nothing but formula (2.1) written in another form.

Let $U$ be the representation of the group $G$ by translations in functions on $G/H$ (quasiregular representation), for example, on the space $C^\infty(G/H)$, and $U$ the corresponding representation of the Lie algebra $\mathfrak{g}$. The correspondence $D \mapsto F$, assigning to an operator its covariant symbol, is $\mathfrak{g}$-equivariant, it means that if $F$ is the covariant symbol of an operator $D = \pi_\lambda^-(X)$, $X \in \mathrm{Env}\,(\mathfrak{g})$, then $U(L)F$, where $L \in \mathfrak{g}$, is the covariant symbol of the operator $\pi_\lambda^-(\mathrm{ad}\,L\cdot X)$.

The multiplication of operators gives rise to the multiplication of covariant symbols, denote it by $*$. Namely, let $F_1$, $F_2$ be covariant symbols of operators $D_1$, $D_2$ respectively. Then the covariant symbol $F_1 * F_2$ of the product $D_1 D_2$ is

$$(F_1 * F_2)(\xi,\eta) = \frac{1}{\Phi(\xi,\eta)}\,(D_1\otimes 1)\left(\Phi(\xi,\eta)F_2(\xi,\eta)\right) .$$

Putting in (3.1) $D = D_1$, $F = F_1$ and $\varphi(u) = \Phi(u,\eta)F_2(u,\eta)$, we get

$$(F_1 * F_2)(\xi,\eta) = \int_{G/H} F_1(\xi,v)F_2(u,\eta)\,\mathcal{B}(\xi,\eta;u,v)\,dx(u,v), \tag{3.3}$$

where

$$\mathcal{B}(\xi,\eta;u,v) = c\,\frac{\Phi(\xi,v)\Phi(u,\eta)}{\Phi(\xi,\eta)\Phi(u,v)} .$$

Let us call this function $\mathcal{B}$ the *Berezin kernel.* It can be regarded as a function $\mathcal{B}(x,y)$ on $G/H \times$ $\times\ G/H$. It is invariant with respect to $G$:

$$\mathcal{B}(\mathrm{Ad}\,g\cdot x,\ \mathrm{Ad}\,g\cdot y) = \mathcal{B}(x,y).$$

Define a transform $x \mapsto x^{\smile}$ of the space $G/H$, that in horospherical coordinates $\xi, \eta$ is the permutation $\xi \leftrightarrow \eta$, i. e. $(\xi, \eta) \mapsto (\eta, \xi)$. This transform induces the transform $F \mapsto F^{\smile}$ of functions in $S(G/H)$: $F^{\smile}(\xi, \eta) = F(\eta, \xi)$. The Berezin kernel is invariant with respect to the simultaneous permutation $\xi \leftrightarrow \eta$ and $u \leftrightarrow v$. By (3.3) it implies that the transform $F \mapsto F^{\smile}$ is an anti-involution with respect to the multiplication of symbols:

$$(F_1 * F_2)^{\smile} = F_2{}^{\smile} * F_1{}^{\smile}.$$

For operators $D$, the transform $F \mapsto F^{\smile}$ of symbols means the conjugation $D \mapsto D^{\smile}$ with respect to the bilinear form generated by the operator $A_\lambda$:

$$(A_\lambda \varphi, \psi) = \int |N(\xi, \eta)|^{-\lambda - \varkappa} \varphi(\xi) \psi(\eta) \, d\xi \, d\eta.$$

Moreover, if $D = \pi_\lambda^-(X)$, then

$$D^{\smile} = \pi_\lambda^+(X^\vee),$$

where $X \mapsto X^\vee$ is the transform of the algebra $\mathrm{Env}(\mathfrak{g})$ induced by the transform $g \mapsto g^{-1}$ in the group $G$.

Thus, the space $\mathcal{A}_\lambda$ with multiplication $*$ is an associative algebra with 1, the transform $F \mapsto F^{\smile}$ is an anti-involution of this algebra.

Now we define *contravariant symbols*. A function (a polynomial) $F(\xi, \eta)$ is the contravariant symbols for the following operator $A$ (acting on functions $\varphi(\xi)$):

$$(A\varphi)(\xi) = c \int_{G/H} F(u, v) \, \frac{\Phi(\xi, v)}{\Phi(u, v)} \, \varphi(u) \, dx(u, v) \tag{3.4}$$

(notice that (3.4) differs from (3.2) by the first argument of $F$ only). This operator is a Töplitz type operator.

Thus, we have two maps: $D \mapsto F$ ("co") and $F \mapsto A$ ("contra"), connecting polynomials on $G/H$ and operators acting on functions $\varphi(\xi)$.

If a polynomial $F$ on $G/H$ is the covariant symbol of an operator $D = \pi_\lambda^-(X)$, $X \in \mathrm{Env}(\mathfrak{g})$, and the contravariant symbol of an operator $A$ simultaneously, then $A = \pi^-_{-\lambda-\varkappa}(X^\vee)$. Therefore, $A$ is obtained from $D$ by the conjugation with respect to the bilinear form

$$(F, f) = \int_{\mathfrak{q}^-} F(\xi) \, f(\xi) \, d\xi.$$

In terms of kernels, it means that the kernel $L(\xi, u)$ of the operator $A$ is obtained from the kernel $K(\xi, u)$ of the operator $D$ by the transposition of arguments and the change $\lambda \mapsto -\lambda - \varkappa$. So, the composition $\mathcal{O}: D \mapsto A$ ("contra" $\circ$ "co") is

$$\mathcal{O}: \; \pi_\lambda^-(X) \longmapsto \pi^-_{-\lambda-\varkappa}(X^\vee).$$

This map commutes with the adjoint representation. Such a map was absent in Berezin's theory for Hermitian symmetric spaces.

The composition $\mathcal{B}$ ("co" $\circ$ "contra") maps the contravariant symbol of an operator $D$ to its covariant symbol. Let us call $\mathcal{B}$ the *Berezin transform*. The kernel of this transform is just the Berezin kernel.

Main problems here are the following. One has to do explicitly: (a) to express the Berezin transform $\mathcal{B}$ in terms of Laplacians $\Delta_1, \ldots, \Delta_r$ ($r$ being the rank), in fact, it is the same that to decompose a canonical representation (see § 4) into irreducible constituents; (b) to compute eigenvalues of $\mathcal{B}$ on irreducible constituents; (c) to find a full asymptotics of $\mathcal{B}$ when $\lambda \to -\infty$ (an analogue of the Planck constant is $h = -1/\lambda$). In particular, two first terms of asymptotic decomposition the Berezin transform $\mathcal{B}$ when $\lambda \to -\infty$ should be

$$\mathcal{B} \sim 1 - \frac{1}{\lambda}\,\Delta,$$

where $\Delta$ is the Laplace–Beltrami operator. It gives the correspondence principle:

$$\begin{aligned} F_1 * F_2 &\quad\longrightarrow\quad F_1 F_2, \\ -\lambda\,(F_1 * F_2 - F_2 * F_1) &\quad\longrightarrow\quad \{F_1,\, F_2\}, \end{aligned}$$

in right hand sides the pointwise multiplication and the Poisson bracket stand.

These problems are solved for spaces of rank one and for spaces with the group $G = \mathrm{SO}_0(p,q)$, for latter spaces the rank is equal to 2, see [7], [8].

## § 4. Canonical representations and quantization

The main tool for studying quantization is the so-called *canonical representations* (this term was introduced in [9]). For Hermitian symmetric spaces $G/K$, they were introduced by Vershik, Gelfand and Graev [9] (for the Lobachevsky plane) and Berezin [1], [2] (in classical case). These representations act by translations in functions on $G/K$ and are unitary with respect to some non-local inner product (now called a *Berezin form*).

We define canonical representations of a group $G$ in a more general setting. We give up the condition of unitarity (as too narrow) and let these representations act on sufficiently extensive spaces, in particular, on spaces of distributions. Moreover, we permit also non-transitive actions of a group $G$. Our approach uses the notion of an "overgroup" and consists in the following.

Let $G$ and $\widetilde{G}$ be semisimple Lie groups such that $G$ is a spherical subgroup of the $\widetilde{G}$ (i. e. $G$ is the fixed point subgroup of an involution of $\widetilde{G}$). We call $\widetilde{G}$ an "overgroup" for $G$. Let $\widetilde{P}$ be a maximal parabolic subgroup of $\widetilde{G}$, such that $\widetilde{P} \cap G = H$. Let $\widetilde{R}_\lambda$, $\lambda \in \mathbb{C}$, be a series of representations of $\widetilde{G}$ induced by characters of $\widetilde{P}$. They can depend on some discrete parameters, we do not write them. As a rule, representations $\widetilde{R}_\lambda$ are irreducible. They act on a compact manifold $\Omega$ (a flag manifold for $\widetilde{G}$). The series $R_\lambda$ has an intertwining operator $Q_\lambda$. Restrictions $R_\lambda$ of $\widetilde{R}_\lambda$ to $G$ are called canonical representations of $G$. They act on functions on $\Omega$. In general, $\Omega$ is not a homogeneous space for $G$, there are several open $G$-orbits on $\Omega$. They are semisimple symmetric spaces $G/H_i$. The manifold $\Omega$ is the closure of the union of open orbits. The intertwining operator $Q_\lambda$ called the Berezin transform.

One can consider a some different version of canonical representations, namely, the restriction of canonical representations in the first sense (on functions on $\Omega$) to some open orbit $G/H_i$. Both variants deserve to be investigated. The first variant is in some sense more natural. But for quantization we need just the second variant.

Recall, see § 1, classification (a), (b), (c), (d) for symplectic symmetric spaces $G/H$. As an overgroup $\widetilde{G}$ for $G$ for classes (a), (b) we take the complexification $G^{\mathbb{C}}$ of $G$, and for classes (c), (d) we take the direct product $G \times G$.

## § 5. Polynomial quantization and the overgroup

Now let $G/H$ be a para-Hermitian symmetric space, see § 2. As an overgroup for $G$ we take the direct product $\widetilde{G}=G\times G$. It contains $G$ as the diagonal $\{(g,g),g\in G\}$.

First we describe a series of representations $\widetilde{R}_\lambda$ of $\widetilde{G}$.

Let $\widetilde{P}$ be a parabolic subgroup $\widetilde{P}$ consisting of pairs $(zh,hn)$, $z\in Q^-$, $h\in H$, $n\in Q^+$. Let $\widetilde{\omega}_\lambda$ be a character of $\widetilde{P}$ equal to $\omega_\lambda(h)$ at these pairs. The representation of $\widetilde{G}$ induced by the character $\widetilde{\omega}_\lambda$ of the subgroup $\widetilde{P}$ is denoted $\widetilde{R}_\lambda$.

Let us give some realizations of representations $\widetilde{R}_\lambda$.

Denote by $\mathcal{C}$ the manifold of "double" cosets

$$y=s_1^{-1}Q^-Q^+s_2,\quad s_1,s_2\in G.$$

This manifold is an analogue of a cone for representations of the pseudo-orthogonal group associated with a cone. The group $\widetilde{G}$ acts on $\mathcal{C}$ as follows:

$$y\mapsto g_1^{-1}yg_2,\quad g_1,g_2\in G. \tag{5.1}$$

Denote by $\mathcal{D}_\lambda(\mathcal{C})$ the space of functions $f$ on $\mathcal{C}$ of class $C^\infty$ satisfying the following homogeneity condition

$$f(s_1^{-1}\,h\,Q^-Q^+s_2)=\omega_\lambda(h)\,f(s_1^{-1}\,Q^-Q^+s_2). \tag{5.2}$$

The representation $\widetilde{R}_\lambda$ acts on $\mathcal{D}_\lambda(\mathcal{C})$ by

$$\big(\widetilde{R}_\lambda(g_1,\,g_2)\,f\big)(y)=f(g_1^{-1}yg_2),\quad g_1,g_2\in G.$$

Let us take in $\mathcal{C}$ two sections: "hyperbolic" section $\mathcal{X}$ and "parabolic" section $\Gamma$.

The manifold $\mathcal{X}\subset\mathcal{C}$ consists of cosets

$$x=s^{-1}Q^-Q^+s,\quad s\in G.$$

The group $G$ acts on $\mathcal{X}$ by $x\mapsto g^{-1}xg$. The stabilizer of the initial point $x^0=Q^-Q^+$ is $H$, so that $\mathcal{X}$ can be identified with $G/H$.

The manifold $\Gamma\subset\mathcal{C}$ consists of cosets

$$\gamma=\exp{(-\eta)}\,Q^-Q^+\exp\xi,\quad \xi\in\mathfrak{q}^-,\ \eta\in\mathfrak{q}^+. \tag{5.3}$$

This manifold can be identified with $\mathfrak{q}^-\times\mathfrak{q}^+$.

Let us embed $\Gamma\hookrightarrow\mathcal{X}$. It is the embedding $\mathfrak{q}^-\times\mathfrak{q}^+\hookrightarrow G/H$, see (1.8), (1.9), (1.10) and further, in terms of $\mathcal{C}$.

Let a point $x=s^{-1}Q^-Q^+s$, $s\in G$, has horospherical coordinates $\xi,\eta$. By (1.9) we find the element $h(\xi,\eta)$:

$$\exp\xi\cdot\exp{(-\eta)}=\exp{(-Y)}\cdot\exp X\cdot h_0,\quad h_0=h(\xi,\eta),$$

where $X\in\mathfrak{q}^-$, $Y\in\mathfrak{q}^+$, and by (1.10) we obtain

$$s=\exp Y\cdot\exp\xi=\exp X\cdot h_0\cdot\exp\eta, \tag{5.4}$$

so that

$$\begin{aligned} x&=s^{-1}Q^-Q^+s\\ &=\exp{(-\eta)}\cdot h_0^{-1}Q^-Q^+\exp\xi. \end{aligned} \tag{5.5}$$

Thus, the embedding above assigns to a point $\gamma \in \Gamma$, given by (5.3), the point $x \in \mathcal{X}$, given by (5.5) where $h_0 = h(\xi, \eta)$.

The representation $\widetilde{R}_\lambda$ can be realized in functions on these manifolds $\mathcal{X}$ and $\Gamma$.

First consider $\mathcal{X}$. A point $x = s^{-1}Q^-Q^+s$ in $\mathcal{X}$ under action (5.1) goes to the point $g_1^{-1}xg_2 = g_1^{-1}s^{-1}Q^-Q^+sg_2$ in $\mathcal{C}$. Take the element $sg_2(sg_1)^{-1}$, i. e. the element $sg_2g_1^{-1}s^{-1}$, and decompose it "by Gauss":

$$sg_2g_1^{-1}s^{-1} = \exp(-Y^*) \cdot \exp X^* \cdot h^*, \;\; X^* \in \mathfrak{q}^-, \;\; Y^* \in \mathfrak{q}^+. \tag{5.6}$$

Here the element $h^* \in H$ depends on the point $x$ only and does not depend on its representative $s$. Let us form an element $s^* \in G$:

$$s^* = \exp Y^* \cdot sg_2 = \exp X^* \cdot h^* sg_1. \tag{5.7}$$

It gives the point $x^* \in \mathcal{X}$:

$$x^* = (s^*)^{-1}Q^-Q^+s^*.$$

By (5.7) we have

$$x^* = g_1^{-1}s^{-1}(h^*)^{-1}Q^-Q^+sg_2.$$

Therefore,

$$f(x^*) = \omega_\lambda((h^*)^{-1})\, f(g_1^{-1}xg_2),$$

so that $\widetilde{R}_\lambda$ acts in functions on $\mathcal{X} = G/H$ as follows:

$$\big(\widetilde{R}_\lambda\,(g_1, g_2)\, f\big)(x) = \omega_\lambda(h^*)\, f(x^*). \tag{5.8}$$

**Theorem 5.1.** *In horospherical coordinates* $\xi, \eta$ *on* $G/H$ *the representation* $\widetilde{R}_\lambda$ *is*

$$\big(\widetilde{R}_\lambda\,(g_1, g_2)\, f\big)(\xi, \eta) = \frac{\Phi_\lambda(\xi \bullet g_2, \eta \circ g_1)}{\Phi_\lambda(\xi, \eta)}\; \omega_\lambda(\widetilde{h}_2)\; \omega_\lambda(\widehat{h}_1^{-1})\; f(\xi \bullet g_2,\, \eta \circ g_1), \tag{5.9}$$

*where* $\widetilde{h}_2$ *and* $\widehat{h}_1$ *are taken from decompositions* (1.6) *and* (1.7) *with* $g = g_2$ *and* $g = g_1$ *respectively.*

**Proof.** Let a point $x = s^{-1}Q^-Q^+s$, $s \in G$, has horospherical coordinates $\xi, \eta$. By (5.4) and (1.6), (1.7) we have

$$sg_2 = \exp Y \cdot \exp \xi \, \cdot g_2 = \exp Y \cdot \exp Y_2 \, \cdot \widetilde{h}_2 \cdot \exp \widetilde{\xi}_2,$$
$$sg_1 = \exp X \cdot h_0 \cdot \exp \eta \, \cdot g_1 = \exp X \cdot h_0 \cdot \exp X_1 \cdot \widehat{h}_1 \cdot \exp \widehat{\eta}_1.$$

where $\widetilde{\xi}_2 = \xi \bullet g_2$, $\widehat{\eta}_1 = \eta \circ g_1$. Hence

$$s^* = \exp Y^* \cdot sg_2 = \exp Y_3 \, \cdot \widetilde{h}_2 \cdot \exp \widetilde{\xi}_2, \tag{5.10}$$

$$s^* = \exp X^* \cdot sg_1 = \exp X_3 \cdot h^* \cdot h_0 \cdot \widehat{h}_1 \cdot \exp \widehat{\eta}_1. \tag{5.11}$$

Therefore, using (5.10) and (5.11), we obtain

$$\begin{aligned} x^* &= (s^*)^{-1}Q^-Q^+s^* \\ &= \exp \widehat{\eta}_1 \cdot \left(h^* h_0 \widehat{h}_1\right)^{-1} \cdot Q^-Q^+ \cdot \widetilde{h}_2 \cdot \exp \widetilde{\xi}_2 \\ &= \exp \widehat{\eta}_1 \cdot \left(h^* h_0 \widehat{h}_1\right)^{-1} \cdot \widetilde{h}_2 \cdot Q^-Q^+ \cdot \exp \widetilde{\xi}_2. \end{aligned}$$

By homogeneity condition (5.2) we have

$$f(x^*) = f\left(\exp \widehat{\eta}_1 \cdot Q^-Q^+ \cdot \exp \widetilde{\xi}_2\right) \cdot \omega_\lambda\left((h^* h_0 \widehat{h}_1)^{-1} \widetilde{h}_2\right) \tag{5.12}$$

On the other hand, by (5.5) we can write the point $x^*$ in the following form:

$$x^* = \exp\widehat{\eta}_1 \cdot (h_0^*)^{-1} Q^- Q^+ \cdot \exp\widetilde{\xi}_2,$$

where $h_0^* = h\left(\widetilde{\xi}_2, \widehat{\eta}_1\right)$. Whence again by homogeneity condition (5.2) we obtain

$$f(x^*) = f\left(\exp\widehat{\eta}_1 \cdot Q^- Q^+ \cdot \exp\widetilde{\xi}_2\right) \cdot \omega_\lambda\left((h_0^*)^{-1}\right). \tag{5.13}$$

Comparing (5.12) and (5.13) we get

$$\omega_\lambda\left(\widehat{h}_1^{-1} h_0^{-1} (h^*)^{-1} \widetilde{h}_2\right) = \omega_\lambda\left((h_0^*)^{-1}\right),$$

whence

$$\omega_\lambda(h^*) = \frac{\omega_\lambda(h_0^*)}{\omega_\lambda(h_0)}\, \omega_\lambda(\widehat{h}_1^{-1})\, \omega_\lambda(\widetilde{h}_2).$$

Substitute it to (5.8) and remember (2.2) and (3.1), as result we obtain (5.9). □

Similarly, the representation $\widetilde{R}_\lambda$ can be realized in functions on the manifold $\Gamma$, it is given by:

$$\left(\widetilde{R}_\lambda(g_1, g_2)\, f\right)(\xi, \eta) = \omega_\lambda(\widetilde{h}_2)\, \omega_\lambda(\widehat{h}_1^{-1})\, f(\xi \bullet g_2, \eta \circ g_1).$$

It shows that $\widetilde{R}_\lambda$ is equivalent to a tensor product:

$$\widetilde{R}_\lambda(g_1, g_2) = \pi_\lambda^-(g_2) \otimes \pi_\lambda^+(g_1).$$

The group $\widetilde{G}$ contains three subgroups isomorphic to $G$. The first one is the diagonal consisting of pairs $(g, g)$, $g \in G$. The restriction of the representation $\widetilde{R}_\lambda$ to this subgroup is the representation $U$ by translations on $G/H$:

$$\left(\widetilde{R}_\lambda(g, g)\, f\right)(x) = f(g^{-1} x g).$$

Indeed, (5.6) and (5.7) with $g_1 = g_2 = g$ give $h^* = e$ and $s^* = sg$.

Two other subgroups $G_1$ and $G_2$ consist of pairs $(g, e)$ and $(e, g)$, where $g \in G$, respectively.

By virtue of Theorem 5.1, the restriction of the representation $\widetilde{R}_\lambda$ to the subgroup $G_2$ is given by

$$\begin{aligned}\left(\widetilde{R}_\lambda(e, g)\, f\right)(\xi, \eta) &= \frac{\Phi_\lambda(\widetilde{\xi}, \eta)}{\Phi_\lambda(\xi, \eta)}\, \omega_\lambda(\widetilde{h})\, f(\widetilde{\xi}, \eta) \\ &= \frac{1}{\Phi_\lambda(\xi, \eta)} \left(\pi_\lambda^-(g) \otimes 1\right) \left[f(\xi, \eta) \Phi_\lambda(\xi, \eta)\right].\end{aligned}$$

Similarly, the restriction of the representation $\widetilde{R}_\lambda$ to the subgroup $G_1$ is given by

$$\left(\widetilde{R}_\lambda(g, e)\, f\right)(\xi, \eta) = \frac{1}{\Phi_\lambda(\xi, \eta)} \left(1 \otimes \pi_\lambda^+(g)\right) \left[f(\xi, \eta) \Phi_\lambda(\xi, \eta)\right].$$

Let us go from the group $G$ to the universal enveloping algebra $\mathrm{Env}(\mathfrak{g})$ and preserve notations for representations. Let us take as $f$ the function $f_0$ equal to the 1 identically. Then for $X \in \mathrm{Env}(\mathfrak{g})$ we have

$$(\widetilde{R}_\lambda(0, X) f_0)(\xi, \eta) = \frac{1}{\Phi_\lambda(\xi, \eta)} (\pi_\lambda^-(X) \otimes 1) \Phi_\lambda(\xi, \eta), \tag{5.14}$$

$$(\widetilde{R}_{-\lambda-\varkappa}(X,0)f_0)(\xi,\eta) = \frac{1}{\Phi_\lambda(\xi,\eta)}(1\otimes\pi^+_{-\lambda-\varkappa}(X))\Phi_{-\lambda-\varkappa}(\xi,\eta). \tag{5.15}$$

Right hand sides of formulae (5.14) and (5.15) are just covariant and contravariant symbols of operator $D=\pi^-_\lambda(X)$ in polynomial quantization, see § 3.

Let us change the position of arguments in $\widetilde{R}_\lambda$, then we have a new representation $\widehat{R}_\lambda$ of $\widetilde{G}$, namely, $\widehat{R}_\lambda(g_1,g_2)=\widetilde{R}_\lambda(g_2,g_1)$. Using the realization of $\widetilde{R}_\lambda$ on the section $\Gamma$, we see that the tensor product $A_\lambda\otimes B_\lambda$ intertwines the representation $\widetilde{R}_\lambda$ with the representation $\widehat{R}_{-\lambda-\varkappa}$. Passing from $\Gamma$ to $\mathcal{X}$ and replacing $\lambda$ by $-\lambda-\varkappa$, we obtain that the operator $c(\lambda)A_{-\lambda-\varkappa}\otimes B_{-\lambda-\varkappa}$ intertwines the representation $\widehat{\widetilde{R}}_{-\lambda-\varkappa}$ with the representation $\widetilde{R}_\lambda$ and transfers contravariant symbols to covariant ones. It has the kernel $\mathcal{B}_\lambda(\xi,\eta;u,v)$, i. e. it is precisely the Berezin transform.

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# КВАНТОВАНИЕ ПО БЕРЕЗИНУ КАК ЧАСТЬ ТЕОРИИ ПРЕДСТАВЛЕНИЙ

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Мы предлагаем новый подход к полиномиальному квантованию (варианту квантования в духе Березина) на параэрмитовых симметрических пространствах с использованием понятия "надгруппы". Этот подход дает ковариантные и контравариантные символы и преобразование весьма естественным и прозрачным способом.
*Ключевые слова:* симплектические многообразия; исчисление символов; квантование; преобразование Березина

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