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ПОЛИНОМИАЛЬНЫЕ АВТОМОРФИЗМЫ, КВАНТОВАНИЕ И ЗАДАЧИ ВОКРУГ ГИПОТЕЗЫ ЯКОБИАНА. II. ДОКАЗАТЕЛЬСТВО КВАНТОВАНИЯ ТЕОРЕМЫ БЕРГМАНА О ЦЕНТРАЛИЗАТОРЕ

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Посвящается памяти Евгения Соломоновича Голода

Аннотация. Целью данного обзора является систематизация результатов, касающихся квантового подхода к некоторым классическим аспектам некоммутативных алгебр, особенно к гипотезе о якобиане. Работа начинается с квантования доказательства теоремы Бергмана о централизации, затем обсуждаются автоморфизмы автоморфизмов INd-схем и вопросы аппроксимации. Последняя глава посвящена связи между теоремами типа Бернсайда теории PI и гипотезой Якоби (подход Ягжева). В данном выпуске публикуется вторая часть работы. Первая часть: Итоги науки и техники. Современная математика и ее приложения. Тематические обзоры. — 2022. — *213*. — С. 110–144. Продолжение будет опубликовано в следующих выпусках.

***Ключевые слова*:** автоморфизм, квантование, гипотеза о Якобиане.

POLYNOMIAL AUTOMORPHISMS, QUANTIZATION, AND JACOBIAN CONJECTURE RELATED PROBLEMS. II. QUANTIZATION PROOF OF BERGMAN'S CENTRALIZER THEOREM

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Abstract. The purpose of this review is the collection and systematization of results concerning the quantization approach to the some classical aspects of non-commutative algebras, especially to the Jacobian conjecture. We start with quantization proof of Bergman centralizing theorem, then discourse authomorphisms of INd-schemes authomorphisms, then go to aproximation issues. Last chapter dedicated to relations between PI-theory Burnside type theorems and Jacobian Conjecture (Jagzev approach). This issue contains the second part of the work. The first part is: Itogi Nauki Tekhn. Sovr. Mat. Prilozh. Temat. Obzory, **213** (2022), pp. 110–144. Continuation will be published in future issues.

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Chapter 2

QUANTIZATION PROOF OF BERGMAN'S CENTRALIZER THEOREM

We first give a brief summary of the background of two well-known centralizer theorems in the power series ring and in the free associative algebra, i.e., Cohn's centralizer theorem and Bergman's centralizer theorem.

2.1. Centralizer Theorems

This section is a relatively independent part of the paper, and only sketches proofs with classic tools, while the following sections will focus on the new proof of Bergman's centralizer theorem.

Throughout this section, X is a finite set of noncommutative variables, and k is a field. Let X^* denote the free monoid generated by X. Let $k\langle X\rangle$ and $k\langle X\rangle$ denote the k-algebra of formal series and noncommutative polynomials (i.e., the free associative algebra over k) in X, respectively. Both elements of $k\langle\langle X\rangle\rangle$ and $k\langle X\rangle$ have the form

$$a = \sum_{\omega\in X^*} a_\omega \omega,$$

where $a_\omega \in k$ is the coefficient of the word w in a, but they have different details inside the above formula. An element of $k\langle X\rangle$ is only a finite sum of words, while there are infinitely many terms of the sum for an element in $k\langle\langle X\rangle\rangle$. The multiplication of elements in $k\langle\langle X\rangle\rangle$ is the concatenation of words and normal multiplication of coefficients. We can only combine the coefficients which have the same corresponding words for addition. The *length* $|\omega|$ of a word $\omega \in X^*$ is the number of letters inside ω.

Now we can define the valuation

$$\nu : k\langle\langle X\rangle\rangle \to \mathbb{N}\cup\{\infty\}$$

as follows: $\nu(0) = \infty$ and if $a = \sum\limits_{\omega\in X^*} a_\omega\omega \neq 0$, then $\nu(a) = \min\{|\omega| : a_\omega \neq 0\}$. Note that if ω is constant, then $\nu(\omega) = 0$ and $\nu(ab) = \nu(a) + \nu(b)$ for all a, b in $k\langle\langle X\rangle\rangle$.

For the words valuation, there is an easy but quite useful lemma [178].

Lemma 2.1.1 (Levi's Lemma)**.** *Let $\omega_1, \omega_2, \omega_3, \omega_4 \in X^*$ be nonzero with $|\omega_2| \geqslant |\omega_4|$. If $\omega_1\omega_2 = \omega_3\omega_4$, then $\omega_2 = \omega\omega_4$ for some $\omega \in X^*$.*

The proof is trivial by backward induction on $|\omega_2|$ since ω_2 has the same last letter as ω_4. Next lemma extends Levi's lemma to $k\langle\langle X\rangle\rangle$, and we post the result as follows.

Lemma 2.1.2 (see [139, Lemma 9.1.2])**.** *Let $a, b, c, d \in k\langle\langle X\rangle\rangle$ be nonzero. If $\nu(a) \geqslant \nu(c)$ and $ab = cd$, then $a = cq$ for some $q \in k\langle\langle X\rangle\rangle$.*

Proof. We can fix a word u which appears in b and $|u| = \nu(b)$. Suppose v is any nonzero word appearing in d, then we have

$$|v| \geqslant \nu(d) = \nu(a) + \nu(b) - \nu(c) \geqslant \nu(b) = |u|. \tag{2.1.1}$$

Let w be any word in X^*. The coefficient of wu in ab is $\sum_{rs=wu} a_r b_s$, where a_r and b_s are the coefficients of the words r, s which appear in a, b respectively. Similarly, the coefficient of wu in cd is $\sum_{yz=wu} c_y d_z$. Since $ab = cd$, we have

$$\sum_{rs=wu} a_r b_s = \sum_{yz=wu} c_y d_z. \tag{2.1.2}$$

By the inequality 2.1.1, we have $|z| \geqslant |u|$, and $|s| \geqslant |u|$ by the definition of u. Thus $rs = wu$ and $yz = wu$ imply $s = s_1 u$ and $z = z_1 u$ for some $s_1, z_1 \in X^*$, by Levi's lemma. Hence $rs_1 = yz_1 = w$ and we can rewrite the formula 2.1.2 as

$$\sum_{rs_1=w} a_r b_{s_1 u} = \sum_{yz_1=w} c_y d_{z_1 u}. \tag{2.1.3}$$

Let

$$b' = \sum_{s_1 \in X^*} b_{s_1 u} s_1, \quad d' = \sum_{z_1 \in X^*} b_{z_1 u} z_1.$$

Then the equation gives $ab' = cd'$. The constant term of b' is $b_u \neq 0$ and hence b' is invertible in $k\langle\langle X\rangle\rangle$. Hence if we let $q = d'b'^{-1}$, then $a = cq$. □

2.1.1. Cohn's centralizer theorem. With the help of the preceding lemmas, we could post and prove this well-known centralizer theorem of k-algebra of formal series by P. M. Cohn.

Theorem 2.1.3 (Cohn's Centralizer Theorem, [62])**.** *If $a \in k\langle\langle X\rangle\rangle$ is not a constant, then the centralizer $C(a; k\langle\langle X\rangle\rangle) \cong k[\![x]\!]$, where $k[\![x]\!]$ is the algebra of formal power series in the variable x.*

Proof. Let $C := C(a; k\langle\langle X\rangle\rangle)$. Let a_0 be the constant term of a, then it is clear that $C = C(a - a_0; k\langle\langle X\rangle\rangle)$. So we may assume that the constant term of a is zero. Thus we have a nonempty set $A = \{c \in C : \nu(c) > 0\}$ because $a \in C$ and so there exists $b \in A$ such that $\nu(b)$ is minimal. An easy observation is that $k[\![b]\!] \cong k[\![x]\!]$. Because suppose $\sum_{i \geqslant m} \beta_i b^i = 0, \beta_i \in k, \beta_m \neq 0$, then we must have $\infty = \nu(\sum_{i \geqslant m} \beta_i b^i) = \nu(b^m) = m\nu(b)$, which is absurd. So we just need to show that $C = k[\![b]\!]$. Assume that an element $c \in C$ is not constant. Our first claim is that there exist $\beta_i \in k$ such that

$$\nu(c - \sum_{i=0}^{n} \beta_i b^i) \geqslant (n+1)\nu(b). \tag{2.1.4}$$

The proof is by induction on n. let β_0 be the constant term of c. Then $c - \beta_0 \in A$ and thus $\nu(c - \beta_0) \geqslant \nu(b)$, by the minimality of b. This proves the $n = 0$ case for the inequality 2.1.4.

Now we need the second claim to complete this induction proof. Our second claim is following: suppose that the constant term of an element $a \in k\langle\langle X\rangle\rangle$ is zero and $b, c \in C \setminus \{0\}$. If $\nu(c) \geqslant \nu(b)$, then $c = bd$ for some $d \in C$. In fact, since the constant term of an element $a \in k\langle\langle X\rangle\rangle$ is zero we have $\nu(a) \geqslant 1$. Thus for n large enough, we have $\nu(a^n) = n\nu(a) \geqslant \nu(c)$. we also have $a^n c = ca^n$ because $c \in C$. Thus, by lemma 2.1.2, $a^n = cq$ for some $q \in k\langle\langle X\rangle\rangle$. Hence, $cqb = a^n b = ba^n$ and since $\nu(c) \geqslant \nu(b)$, we have $c = bd$, for some $d \in k\langle\langle X\rangle\rangle$, by lemma 2.1.2. Finally,

$$bad = abd = ac = ca = bda,$$

which gives $ad = da$, i.e. $d \in C$.

Now let us continue to prove the first claim. Suppose we have found $\beta_0, \dots, \beta_n \in k$ such that $\nu(c - \sum_{i=0}^{n} \beta_i b^i) \geqslant (n+1)\nu(b)$. Then since $(n+1)\nu(b) = \nu(b^{n+1})$, we have $c - \sum_{i=0}^{n} \beta_i b^i = b^{n+1} d$ for some $d \in C$, by the second claim we proved above. If d is a constant, we are done because then $c \in k[b] \subset k[\![b]\!]$. Otherwise, let β_{n+1} be the constant term of d. Then $d - \beta_{n+1} \in A$ and hence

$\nu(d-\beta_{n+1}) > \nu(b)$ by the minimality of b. Therefore, by the first claim, $d-\beta_{n+1} = bd'$ for some $d' \in C$. Hence

$$c - \sum_{i=0}^{n} \beta_i b^i = b^{n+1}d = b^{n+1}(bd' + \beta_{n+1}) = b^{n+2}d' + \beta_{n+1}b^{n+1},$$

which gives $c - \sum_{i=0}^{n+1} \beta_i b^i = b^{n+2}d'$. Hence

$$\nu(c - \sum_{i=0}^{n+1} \beta_i b^i) = \nu(b^{n+2}d') = (n+2)\nu(b) + \nu(d') \geqslant (n+2)\nu(b).$$

This completes the induction, then we are done because $\nu(c - \sum_{i\geqslant 0} \beta_i b^i) = \infty$ and so $c = \sum_{i\geqslant 0} \beta_i b^i \in k[\![b]\!]$. □

2.1.2. Bergman's centralizer theorem. Now since $k\langle X\rangle \subset K\langle\langle X\rangle\rangle$, it follows from the above theorem that if $a \in k\langle X\rangle$ is not constant, then $C(a; k\langle X\rangle)$ is commutative because $C(a; k\langle\langle X\rangle\rangle)$ is commutative. The next theorem is our main goal which shows that there is a similar result for $C(a; k\langle X\rangle)$.

Theorem 2.1.4 (Bergman's Centralizer Theorem, [45])**.** *If $a \in k\langle X\rangle$ is not constant, then the centralizer $C(a; k\langle X\rangle) \cong k[x]$, where $k[x]$ is the polynomial algebra in one variable x.*

We will not fully recover the original proof of Bergman's centralizer theorem since this is not our main idea. However, we would take some necessary result in his original proof [45] which helps us to finish the proof of that the centralizer is integrally closed. This will be shown in the Subsection 2.4.3.

First of all, we need to emphasize that the proof of Cohn's centralizer theorem is included. Here is a sketch of the proof.

For simplicity, we denote by $C := C(a; k\langle X\rangle)$ the centralizer of a which from now on is not a constant. Recall that the centralizer C is also commutative. Moreover, C is finitely generated, as module over $k[a]$ or as algebra. Then since $k\langle X\rangle$ is a 2-fir (free ideal rings, cf. Lemma 1.5 in [45]), and the center of a 2-fir is integrally closed, we obtain that the centralizer of a is integrally closed in its field of fractions after using the lifting to $k\langle X\rangle \otimes k(x)$ (where x is a free variable). Then our aim is to show that C is a polynomial ring over k. In order to get this fact we shall study homomorphisms of C into polynomial rings. By using "infinite" words, we obtained an embedding from C into polynomial rings by lexicographically ordered semigroup algebras, which completes this sketch of the proof. Indeed, any subalgebra not equal to k of a polynomial algebra $k[x]$ that is integrally closed in its own field of fractions is of form $k[y]$ (by Lüroth's theorem).

We conclude this section by pointing out that the method of "infinite" words inspires us to find a possibility to prove Bergman's centralizer theorem by deformation quantization. In the next section, we will establish this new approach of quantization for generic matrices.

2.2. Reduction to Generic Matrix

In this section, we will establish an important theorem which gives a relation of commutative subalgebras in the free associative algebra and the algebra of generic matrices. Let $k\langle X\rangle$ be the free associative algebra over a field k generated by a finite set $X = \{x_1, \ldots, x_s\}$ of s indeterminates, and let $k\langle X_1, \ldots, X_s\rangle$ be the algebra of $n \times n$ generic matrices generated by the matrices X_ν. The canonical homomorphism $\pi : k\langle x_1, \ldots, x_s\rangle \to k\langle X_1, \ldots, X_s\rangle$ shows in last section.

We claim that if we have a commutative subalgebra of rank two in the free associative algebra $k\langle X\rangle$, then we also have a commutative subalgebra of rank two if we consider a reduction to generic matrices of big enough order n. We also call two elements of a free algebra *algebraically independent* if the subalgebra generated by these two elements is a free algebra of rank two. Otherwise we will call them *algebraically dependent.*

In other words, if we have a commutative subalgebra $k[f,g]$ of rank two in the free associative algebra, then we have to prove that its projection to generic matrices of some order also has rank two. i.e. $\pi(f), \pi(g)$ do not have any relations.

We need following theorem:

Theorem 2.2.1. *Let $k\langle X\rangle$ be the free associative algebra over a field k generated by a finite set X of indeterminates. If $k\langle X\rangle$ has a commutative subalgebra with two algebraically independent generators $f, g \in k\langle X\rangle$, then the subalgebra of n by n generic matrices generated by reduction of f and g in $k\langle X_1, \ldots, X_s\rangle$ also has rank two for big enough n.*

Proof. Assume $k[f,g]$ be a commutative subalgebra generated by $f, g \in k\langle X\rangle \setminus k$ with rank two. We denote $\bar{f}, \bar{g} \in k\langle X_1, \ldots, X_s\rangle$ to be the generic matrices of f and g respectively after reduction (**??**) of algebra of generic matrices with $n \times n$. The rank of $k\langle \bar{f}, \bar{g}\rangle$ must be $\leqslant 2$ (i.e. it must be 1 or 2). Suppose the rank is 1, then for any two elements $a, b \in k\langle f, g\rangle$, there exists a minimal polynomial $P(x,y) \in k[x,y]$ (x, y are two free variables) with degree m such that $P(\bar{a}, \bar{b}) = 0$ because the algebra of generic matrices is a domain by Theorem **??**. On the other hand, by Amitsur-Levitzki Theorem **??**, there exists no polynomial with degree less than $2n$, such that $P(\bar{a}, \bar{b}) = 0$. This leads to be a contradiction if we choose $n > [m/2]$. □

Recall from the section 2.1.2 that the centralizer $C := C(a; k\langle X\rangle)$ of $a \in k\langle X\rangle \setminus k$ is a commutative subalgebra of $k\langle X\rangle$, so from the above theorem, we conclude that if the centralizer is a subalgebra in $k\langle X\rangle$ of rank two then the π-image subalgebra of C has also rank two.

However, we prefer discussing this general case of subalgebras instead of just consider a centralizer subalgebra. Furthermore, we want to prove that there is no commutative subalgebras of the free associative algebra $k\langle X\rangle$ of rank greater than or equal to two.

2.3. Quantization Proof of Rank One

Up to our knowledge, there is no new proofs has been appeared after Bergman [45] for almost fifty years. We are using a method of deformation quantization presented by M. Kontsevich to give an alternative proof of Bergman's centralizer theorem. In this section [120], we got that the centralizer is a commutative domain of transcendence degree one.

Let $k\langle X\rangle$ be the free associative algebra over a field k generated by s free variables $X = \{x_1, \ldots, x_s\}$. Now, we concentrate on our proof that there is no commutative subalgebras of rank greater than or equal to two. From the homomorphism $\pi : k\langle x_1, \ldots, x_s\rangle \to k\langle X_1, \ldots, X_s\rangle$ and Theorem 2.2.1, we are moving our goal from the elements of $k\langle X\rangle$ to the algebra of generic matrices $k\langle X_1, \ldots, X_s\rangle$, and we consider the quantization of this algebra and its subalgebras.

Let A, B be two commuting generic matrices in $k\langle X_1, \ldots, X_s\rangle$ which are algebraically independent, i.e. $\operatorname{rank} k\langle A, B\rangle = 2$. We have the following theorem.

Theorem 2.3.1. *Let A, B be two commuting generic matrices in $k\langle X_1, \ldots, X_s\rangle$ with $\operatorname{rank} k\langle A, B\rangle = 2$, and let $\hat{A}$ and $\hat{B}$ be quantized images (by sending multiplications to star products by means of Kontsevich's formal quantization) of A and B respectively by considering lifting A and B in $k\langle X_1, \ldots, X_s\rangle[[\hbar]]$. Then $\hat{A}$ and $\hat{B}$ do not commute. Moreover,*

$$\frac{1}{\hbar}[\hat{A}, \hat{B}]_\star \equiv \begin{pmatrix} \frac{1}{\hbar}\{\lambda_1, \mu_1\} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\hbar}\{\lambda_n, \mu_n\} \end{pmatrix} \mod \hbar \tag{2.3.1}$$

where λ_i and μ_i are eigenvalues(weights) of A and B respectively.

To prove this theorem, we need some preparations. It is not easy to directly compute such two generic matrices with order n. However, if we can diagonalize those matrices, then computation will be easier. So first of all, we should show the possibilities. Without loss of generality, we may assume that one of the generic matrices B is diagonal if we have a proper choice of basis of the algebra of generic matrices. Now consider the other generic matrix A which we mentioned above.

Remark 2.3.2. *The generic matrix A may not be diagonalizable over $k[x_{ij}^{(\nu)}]$, but it can be diagonalized over some integral extension of the algebra $k[x_{ij}^{(\nu)}]$ with $i, j = 1, \ldots, n; \nu = 1, \ldots, s$.*

Remark 2.3.3. *Any non-scalar element A of the algebra of generic matrices must have distinct eigenvalues. In fact, by Amitsur's Theorem* **??**, *namely, the algebra of generic matrices is an domain, if the minimal polynomial is not a central polynomial, then the algebra can be embedded to a skew field. Hence, the minimal polynomial is irreducible, and the eigenvalues are pairwise different.*

Lemma 2.3.4. *Let $\hat{A} \equiv A_0 + \mathfrak{h}A_1 (\operatorname{mod} \mathfrak{h}^2)$ be the quantized image of a generic matrix $A \in k\langle X_1, \ldots, X_s\rangle$, where A_0 is diagonal with distinct eigenvalues. Then, the quantized images $\hat{A}$ can be diagonalized over some finite extension of $k[x_{ij}^{(\nu)}]$.*

Proof. Without loss of generality, suppose A_0 is a diagonal generic matrix with distinct eigenvalues. We want to show that there exists an invertible generic matrix P, such that PAP^{-1} is diagonal. Now we consider their images on $k\langle X_1, \ldots, X_s\rangle[[\mathfrak{h}]]$, we may assume $\hat{P} = I + \mathfrak{h}T$ and the conjugation inverse $\hat{P}^{-1} = I - \mathfrak{h}T \mod \mathfrak{h}^2$ (where I is the identity matrix). Then we have

$$(I + \mathfrak{h}T)(A_0 + \mathfrak{h}A_1)(I - \mathfrak{h}T) = A_0 + \mathfrak{h}([T, A_0] + A_1) \mod \mathfrak{h}^2,$$

and we need to solve the equation $[T, A_0] = -A_1$.

This is clear since A_0 is diagonal. Let $A_0 = \operatorname{diag}\{\lambda_1, \ldots, \lambda_n\}$, $T = (t_{ij})_{n\times n}$ and $A_1 = (a_{ij})_{n\times n}$, then we have $[T, A_0] = ((\lambda_i - \lambda_j)\, t_{ij})_{n\times n}$. Hence,

$$T = (t_{ij})_{n\times n} = \left(-\frac{a_{ij}}{\lambda_i - \lambda_j}\right)_{n\times n}.$$

So far, we have determined the $\mathfrak{h}$ term of the matrix $\hat{A}$. Hence, we may assume $\hat{A} \equiv A_0 + \mathfrak{h}^2 A_2 \mod \mathfrak{h}^3$, then we continue to cancel the $\mathfrak{h}^2$ term. Let $\hat{P}_2 = I + \mathfrak{h}^2 T_2$, and the conjugation inverse $\hat{P}_2^{-1} = I - \mathfrak{h}^2 T_2$. Then, we have

$$(I + \mathfrak{h}^2 T_2)(A_0 + \mathfrak{h}^2 A_2)(I - \mathfrak{h}^2 T_2) = A_0 + \mathfrak{h}^2([T_2, A_0] + A_2) \mod \mathfrak{h}^3,$$

Hence, T_2 is determined by equation $[T_2, A_0] = -A_2 = (a_{ij}^{(2)})_{n\times n}$. Similar computation give all entries of T_2, namely

$$T_2 = \left(-\frac{a_{ij}^{(2)}}{\lambda_i - \lambda_j}\right)_{n\times n}.$$

Continue this process to cancel the term of $\mathfrak{h}^3$ etc., we obtain equations $[T_i, A_0] = -A_i$ for $i = 3, 4, 5 \ldots$. This leads to the result that A could be diagonalized over the extension $k[x_{ij}^{(\nu)}][\frac{1}{\lambda_i - \lambda_j}]$. □

Now A, B are two algebraically independent but commuting generic matrices in $k\langle X_1, \ldots, X_s\rangle$. From previous discussion, we may assume A and B can be both diagonalized over an integral extension of $k[x_{ij}^{(\nu)}]$. Consider result of diagonalization in $k\langle X_1, \ldots, X_s\rangle[[\mathfrak{h}]]$ and then we compute the quantization commutator of two quantized generic matrices over $k\langle X_1, \ldots, X_s\rangle[[\mathfrak{h}]]$. Now we can complete the proof of Theorem 2.3.1.

Proof of Theorem 2.3.1. We have shown that A, B can be both diagonalized over some finite extension of $k[x_{ij}^{(\nu)}]$, then consider result of diagonalization with the quantization form in $k\langle X_1, \ldots, X_s\rangle[[\mathfrak{h}]]$, i.e., we can write them into specific forms modulo $\mathfrak{h}^2$ as follows:

$$\hat{A} \equiv \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} + \mathfrak{h} \begin{pmatrix} \delta_1 & & * \\ & \ddots & \\ * & & \delta_n \end{pmatrix} \mod \mathfrak{h}^2$$

$$\hat{B} \equiv \begin{pmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \end{pmatrix} + \mathfrak{h} \begin{pmatrix} \nu_1 & & * \\ & \ddots & \\ * & & \nu_n \end{pmatrix} \mod \mathfrak{h}^2.$$

Then we can compute the quantization commutator,

$$[\hat{A}, \hat{B}]_\star := \hat{A} \star \hat{B} - \hat{B} \star \hat{A} \equiv \begin{pmatrix} \{\lambda_1, \mu_1\} & & 0 \\ & \ddots & \\ 0 & & \{\lambda_n, \mu_n\} \end{pmatrix} + \mathfrak{h}\vec{\lambda} \star \begin{pmatrix} 0 & & * \\ & \ddots & \\ * & & 0 \end{pmatrix} - \mathfrak{h} \begin{pmatrix} 0 & & * \\ & \ddots & \\ * & & 0 \end{pmatrix} \star \vec{\lambda}$$

$$+ \mathfrak{h} \begin{pmatrix} 0 & & * \\ & \ddots & \\ * & & 0 \end{pmatrix} \star \vec{\mu} - \mathfrak{h}\vec{\mu} \star \begin{pmatrix} 0 & & * \\ & \ddots & \\ * & & 0 \end{pmatrix} + \mathfrak{h}^2 \left\{ \begin{pmatrix} 0 & & * \\ & \ddots & \\ * & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & * \\ & \ddots & \\ * & & 0 \end{pmatrix} \right\} \mod \mathfrak{h}^2.$$

Note that all terms have empty diagonals except the first term, and hence the quantization commutator $[\hat{A}, \hat{B}]_\star \neq 0 \mod \mathfrak{h}^2$, which completes the proof of the theorem by multiplying $\frac{1}{\mathfrak{h}}$ on two sides of above equation. □

Remark 2.3.5. *Suppose λ_i and δ_i, $i = 1, \ldots, n$ are algebraically dependent. Then there are polynomials P_i in two variables such that $P_i(\lambda_i, \delta_i) = 0$. Put*

$$P(x, y) = \prod_{i=1}^{n} P_i(x, y).$$

Then $P(A, B)$ is diagonal matrix having zeros on the main diagonal, i.e. $P(A, B) = 0$. It means that if $\operatorname{rank} k\langle A, B\rangle = 2$ then λ_i, δ_i are algebraically independent for some i.

Let us conclude this section by pointing out the whole process of this proof. Recall that we have the free associative algebra $k\langle X\rangle$ over a field k, if we have a commutative subalgebra of rank two generated by $a, b \in k\langle X\rangle$, then we may have a commutative subalgebra of the algebra of generic matrices $k\langle X_1, \ldots, X_s\rangle$ of rank two generated by A, B (they are images of a homomorphism $\pi : k\langle X\rangle \to k\langle X_1, \ldots, X_s\rangle$). Consider the element $0 = [a, b]$ of the free associative algebra $k\langle X\rangle$, homomorphism π and canonical quantization homomorphism q sending multiplications to star products, we yield that

$$0 = q\pi([a, b]) = q[A, B] = [\hat{A}, \hat{B}]_\star.$$

This leads a contradiction to Theorem 2.3.1 which shows that $[\hat{A}, \hat{B}]_\star \neq 0$. So we obtain the following result.

Theorem 2.3.6. *There is no commutative subalgebras of rank $\geqslant 2$ in the free associative algebra $k\langle X\rangle$.* □

The centralizer ring is commutative from our discussion in section 2.1.2, and from the above theorem, it is of rank 1. So it is a commutative subalgebra with form $k[x]$ for some $x \in k\langle X\rangle \setminus k$. We will show it implies Bergman's centralizer theorem 2.1.4 in the next section.

2.4. Centralizers Are Integrally Closed

We have shown that the centralizer C is a commutative domain of transcendence degree one. For us, it was the most interesting part of the proof of the Bergman's centralizer theorem. However, we have to prove the fact that C is integrally closed in order to complete the proof of Bergman's Centralizer Theorem. In our this work [121], our proofs are based on the characteristic free instead of very rich and advanced P. Cohn and G. Bergman's noncommutative divisibility theorem, we use generic matrices reduction, the invariant theory of characteristic zero by C. Procesi [170] and the invariant theory of positive characteristic by A. N. Zubkov [235, 236] and S. Donkin [77, 78].

2.4.1. Invariant theory of generic matrices. We will try to review some useful facts in the invariant theory of generic matrices.

Consider the algebra $\mathbb{A}_{n,s}$ of s-generated generic matrices of order n over the ground field k. Let $a_\ell = (a^\ell_{ij}), 1 \leqslant i, j \leqslant n, 1 \leqslant \ell \leqslant s$ be its generators. Let $R = k[a^\ell_{ij}]$ be the ring of entries coefficients. Consider an action of matrices $M_n(k)$ on matrices in R by conjugation, namely $\varphi_B : B \mapsto MBM^{-1}$. It is well-known (refer to [169,170,235]) that the invariant function on this matrix can be expressed as a polynomial over traces $\operatorname{tr}(a_{i1}, \dots, a_{is})$. Any invariant on $\mathbb{A}_{n,s}$ is a polynomial of $\operatorname{tr}(a_{i1}, \dots, a_{in})$. Note that the conjugation on B induces an automorphism φ_B of the ring R. Namely, $M(a_{ij})^\ell M^{-1} = ({a'_{ij}}^\ell)$, and $\varphi_B(M)$ of R induces automorphism on $M_n(R)$. And for any $x \in \mathbb{A}_{n,s}$, we have

$$\varphi_B(x) = MxM^{-1} = \operatorname{Ad}_M(x).$$

Consider $\varphi_B(x) = \operatorname{Ad}^{-1}_M \varphi_M(x)$. Then any element of the algebra of generic matrices is invariant under $\varphi_M(x)$.

When dealing with matrices in characteristic 0, it is useful to think that they form an algebra with a further unary operation the *trace*, $x \mapsto tr(x)$. One can formalize this as follows [60]:

Definition 2.4.1. *An algebra with trace is an algebra equipped with an additional trace structure, that is a linear map* $\operatorname{tr} : R \to R$ *satisfying the following properties*

$$\operatorname{tr}(ab) = \operatorname{tr}(ba), a \operatorname{tr}(b) = \operatorname{tr}(b)a, \operatorname{tr}(\operatorname{tr}(a)b) = \operatorname{tr}(a)\operatorname{tr}(b) \quad \textit{for all } a, b \in R.$$

There is a well-known fact as follows.

Theorem 2.4.2. *The algebra of generic matrices with trace is an algebra of concomitants, i.e. subalgebra of* $M_n(R)$ *is an invariant under the action* $\varphi_M(x)$.

This theorem was first proved by C. Procesi in [170] for the ground field k of characteristic zero. If k is a field of positive characteristic, we have to use not only traces, but also characteristic polynomials and their linearization (refer to [77, 78]). Relations between these invariants are discovered by C. Procesi [169,170] for characteristic zero and A. N. Zubkov [235,236] for characteristic p. C. de Concini and C. Procesi also generalized a characteristic free approach to invariant theory [71].

Let us denote by $k_T\{X\}$ the algebra of generic matrices with traces. After above discussions, we have the following proposition.

Proposition 2.4.3. *Let* n *be a prime number, then the centralizer of* $A \in k_T\{X\}$ *is rationally closed in* $k_T\{X\}$ *and integrally closed in* $k_T\{X\}$.

2.4.2. Centralizers are integrally closed. Let $k\langle X\rangle$ be the free associative algebra as noted. Here we will prove the following theorem.

Theorem 2.4.4. *The centralizer* C *of non-trivial element* f *in the free associative algebra is integrally closed.*

Let $g, P, Q \in C := C(f; F_z)$, and suppose $gQ^m = P^m$ for some positive integer m, i.e. in localization $g = \frac{P^m}{Q^m}$. Then there exists $h \in C$, such that $h^m = g$. This means that the centralizer C is integral closed.

Consider the homomorphism π from the free associative algebra F_s to the algebra of generic matrices with traces $k_T\{X\}$. Let us denote by $\bar{g}$ the image $\pi(g)$. Then we have following proposition.

Proposition 2.4.5. *Consider the homomorphism* $\pi : F_s \to k_T\{X\}$. *Let the order of matrices be a prime number* $p \gg 0$. $\overline{g} = \pi(g), \overline{P} = \pi(P)$ *and* $\overline{Q} = \pi(Q)$. *Then there exists* $\overline{h} \in k_T\{X\}$ *such that*

(1) $\bar{h}^m = \overline{g}$;
(2) $\bar{h} = \frac{\overline{P}}{\overline{Q}}$;
(3) $\bar{h} \in \overline{C}$, *where* $\overline{C} = \pi(C)$.

Proof. (1) and (2) follows from Proposition 2.4.3 that the algebra of generic matrices with traces of form is integral closed. Prove (3). Note that all eigenvalues of $\bar{g}$ are pairwise different due to Proposition **??**. So is $\bar{f}$. Hence $\bar{f}, \bar{g}$ are diagonalizable and $\bar{h}$ can be diagonalized in the same eigenvectors basis. Hence, by Proposition **??**, $\bar{h}$ commutes with $\bar{f}$, i.e., $\bar{h} \in \overline{C}$. □

Now we have to prove that $\bar{h}$ in fact belongs to the algebra of generic matrices without trace. We use the local isomorphism to get rid of traces.

Definition 2.4.6 (Local isomorphism)**.** *Let $\mathbb{A}$ be an algebra with generators $a_1, \ldots, a_s$ homogeneous respect this set of generators, and let $\mathbb{A}'$ be an algebra with generators $a_1', \ldots, a_s'$ homogeneous respect this set of generators. We say that $\mathbb{A}$ and $\mathbb{A}'$ are locally L-isomorphic if there exist a linear map $\varphi : a_i \to a_i'$ on the space of monomials of degree $\leqslant 2L$, and in this case for any two elements $b_1, b_2 \in A$ with highest term of degree $\leqslant L$, we have*

$$b_i = \sum_j M_{ij}(a_1, \ldots, a_s), b_i' = \sum_j M_{ij}(a_1', \ldots, a_s'),$$

where M_{ij} are monomials, and for $b = b_1 \cdot b_2, b' = b_1' \cdot b_2'$, we have $\varphi(b) = b'$.

We need following lemmas, and propositions:

Lemma 2.4.7 (Local isomorphism lemma)**.** *For any L, if s is big enough prime, then the algebra of generic upper triangular matrices $\mathbb{U}_s$ is locally L-isomorphic to the free associative algebra. Also reduction on the algebra of generic matrices of degree n provides an isomorphism up to degree $\leqslant 2s$.*

Let us remind a well-known and useful fact.

Proposition 2.4.8. *The trace of every element in $\mathbb{U}_s$ of any characteristic is zero.*

In fact, we also proved

Proposition 2.4.9. *If $n > n(L)$, then the algebra of generic matrices (without traces) is L-locally integrally closed.*

Lemma 2.4.10. *Consider the projection $\overline{\pi}$ of the algebra of generic matrices with trace to $\mathbb{U}_s$, sending all traces to zero. Then we have*

$$\overline{\pi}(\overline{h})^m = \overline{\pi}(g).$$

Proof of Theorem 2.4.4. Let p be a big enough prime number. For example, we can set $p \geqslant 2(\deg(f) + \deg(g) + \deg(P) + \deg(Q))$. Because space of $k_T\{X\}$ of degree $\leqslant p$ is isomorphic to space of free associative algebra. We have element h corresponding to $\bar{h}$ up to this isomorphism. Due to local isomorphism, $h^m = g$, $h = P/Q$, i.e. $hQ = P$. Also we have h commutes with f, i.e. $h \in C$. □

2.4.3. Completion of the proof. From last two subsections, we have the following proposition:

Proposition 2.4.11. *Let p be a big enough prime number, and $k\{X\}$ the algebra of generic matrices of order p. For any $A \in k\{X\}$, the centralizer of A is rationally closed and integrally closed in $k\{X\}$ over the center of $k\{X\}$.*

In our previous paper [120], we establish that the centralizer in the algebra of generic matrices is a commutative ring of transcendence degree one. According to Proposition 2.4.11, $C(A)$ is rationally closed and integrally closed in $k\{X\}$. If p is big enough, then $k\{X\}$ is L-locally integrally closed.

Now we need one fact from the Bergman's paper [45]. Let X be a totally ordered set, W be the free semigroup with identity 1 on set X. We have the following lemma.

Lemma 2.4.12 (Bergman)**.** *Let $u, v \in W \setminus \{1\}$. If $u^\infty > v^\infty$, then we have*

$$u^\infty > (uv)^\infty > (vu)^\infty > v^\infty.$$

Proof (*Bergman*). It suffices to show that the whole inequality is implied by $(uv)^\infty > (vu)^\infty$. Suppose $(uv)^\infty > (vu)^\infty$, then we have following

$$(vu)^\infty = v(uv)^\infty > v(vu)^\infty = v^2(uv)^\infty > v^2(vu)^\infty = \cdots v^\infty.$$

Similarly, we obtain $(uv)^\infty < u^\infty$. □

Similarly, we also have inequalities with "⩾" replaced by "=" or "⩽."

Remark 2.4.13. *Similar constructions are used in [113] for Burnside type problems or the height theorem of Shirshov.*

Now let R be the semigroup algebra on W over field k, i.e. $R = F_s$ is the free associative algebra. Consider $z \in \overline{W}$ be an infinite period word, and we denote $R_{(z)}$ be the k-subspace of R generated by words u such that $u = 1$ or $u^\infty \leqslant z$. Let $I_{(z)}$ be the k-subspace spanned by words u such that $u \neq 1$ and $u^\infty < z$. Using Lemma 2.4.12, we can get that $R_{(z)}$ is a subring of R and I_z is a two-sided ideal in $R_{(z)}$. It follows that $R_{(z)}/I_{(z)}$ will be isomorphic to a polynomial ring $k[v]$.

Proposition 2.4.14 (Bergman). *If $C \neq k$ is a finitely generated subalgebra of F_s, then there is a homomorphism f of C in to polynomial algebra over k in one variable, such that $f(C) \neq k$.*

Proof (*Bergman*). First let us totally order X. Let G be a finite set of generators for C and let z be maximum over all monomials $u \neq 1$ with nonzero coefficient in elements of G of u^∞. Then we have $G \subseteq R_{(z)}$ and hence $C \subseteq R_{(z)}$, and the quotient map $f : R_{(z)} \to R_{(z)}/I_{(z)} \cong k[v]$ is nontrivial on C. □

Now we can complete the proof of Bergman's centralizer theorem.

Proof. Consider homomorphism from the Proposition 2.4.14. Because C is centralizer of F_s, it has transcendence degree 1. Consider homomorphism ρ send C to the ring of polynomial. The homomorphism has kernel zero, otherwise $\rho(C)$ will have smaller transcendence degree. Note that C is integrally closed and finitely generated, hence it can be embedded into polynomial ring of one indeterminate. Since C is integrally closed, it is isomorphic to polynomial ring of one indeterminate.

Consider the set of system of C_ℓ, ℓ-generated subring of C such that $C = \cup_\ell C_\ell$. Let $\overline{C_\ell}$ be the integral closure of C_ℓ. Consider set of embedding of C_ℓ to ring of polynomial, then $\overline{C_\ell}$ are integral closure of those images, $\overline{C_\ell} = k[z_\ell]$, where z_ℓ belongs to the integral closure of C_ℓ. Consider sequence of z_ℓ. Because $k[z_\ell] \subseteq k[z_{\ell+1}]$, and degree of $z_{\ell+1}$ is strictly less than the degree of z_ℓ. Hence this sequence stabilizes for some element x. Then $k[z]$ is the needed centralizer. □

2.4.4. On the rationality of subfields of generic matrices. We will discuss some approaches to the following open problem.

Problem 2.4.15. *Consider the algebra of generic matrices $k\{X\}$ of order s. Consider* $\mathrm{Frac}(k\{X\})$, *and K a subfield of* $\mathrm{Frac}(k\{X\})$ *of transcendence degree one over the base field k. Is it true that K is isomorphic to a rational function over k, namely $K \cong k(t)$?*

Let $k\{X\}$ be the algebra of generic matrices of a big enough prime order $s := p$. Let Λ be the diagonal generic matrix $\Lambda = \mathrm{diag}\,(\lambda_1, \dots, \lambda_s)$ in $k\{X\}$, where transcendence degrees satisfy in $\mathrm{Trdeg}\, k[\lambda_i] = 1$. Let N be another generic matrix, whose coefficients are algebraically independent from $\lambda_1, \dots, \lambda_s$. It means that if R is a ring of all coefficients of N, with $\mathrm{Trdeg}(R) = s^2$, then

$$\mathrm{Trdeg}\, R[\lambda_1, \dots, \lambda_s] = s^2 + \mathrm{Trdeg}\, k[\lambda_1, \dots, \lambda_s].$$

Proposition 2.4.16. *We consider the conjugation of generic matrices $\overline{f}$ and $\overline{g}$.*

(a) *Let $k[f_{ij}]$ be a commutative ring and $I = \langle f_{1i} \rangle \lhd k[f_{ij}]$ $(i > 1)$ be an ideal of $k[f_{ij}]$. Then $k[f_{11}] \cap I = 0$.*
(b) *Let $k[f_{ij}, g_{ij}]$ be a commutative ring and $J = \langle f_{1j}, g_{1j} \rangle \lhd k[f_{ij}, g_{ij}]$ $(i, j > 1)$. For any algebraic function P satisfies $P(f_{11}, g_{11}) = 0$, which means f and g algebraically depends on e_1, then $k[f_{11}, g_{11}] \cap J = 0$.*

Corollary 2.4.17. *Let $\mathbb{A}$ be an algebra of generic matrices generated by $a_1, \dots, a_s, a_{s+1}$. Let $f \in k[a_1, \dots, a_s]$, $\varphi = a_{s+1} f a_{s+1}^{-1}$. Let $I = \langle \varphi_{1i} \rangle \lhd k[a_1, \dots, a_{s+1}]$. Then $k[\varphi_{11}] \cap I = 0$.*

Proof. Note that $f = \tau \Lambda \tau^{-1}$ for some τ and a diagonal matrix Λ by proposition 2.4.16. Then $\varphi = (a_{n+1}\tau)\Lambda(a_{n+1}\tau)^{-1}$ and we can treat $(a_{n+1}\tau)$ as a generic matrix. □

Theorem 2.4.18. *Let $C := C(f; F_n)$ be the centralizer ring of $f \in F_n \setminus k$. $\overline{C}$ is the reduction of generic matrices, and $\overline{\overline{C}}$ is the reduction on first eigenvalue action. Then $\overline{\overline{C}} \cong C$.*

Proof. Let us recall that we already have $\overline{C} \cong C$ in [120]. If we have $P(g_1, g_2) = 0$, then clearly $P(\lambda_1(g_1), \lambda_2(g_2)) = 0$ in the reduction on first eigenvalue action. Suppose $\overline{\overline{P(g_1, g_2)}} = 0$. Then $P(g_1, g_2)$ is an element of generic matrices with at least one zero eigenvalue. Because minimal polynomial is irreducible, that implies that $P(g_1, g_2) = 0$. It means any reduction satisfying λ_1 satisfies completely. That what we want to prove. □

Consider $\overline{C}$, for any $\overline{g} = (g_{ij}) \in \overline{C}$. Investigate g_{11}. Suppose there is a polynomial P with coefficients in k, such that $P(f, g) = 0$. We can make a proposition about intersection of the ideals even sharper.

Let $J = \langle f_{1j}, g_{1j} \rangle$ $(j > 1)$ be an ideal of the commutative subalgebra $k[f_{ij}, g_{ij}]$, then $k[f_{11}, g_{11}] \cap J = 0$.

From the discussion above and the theorem 2.4.18, we have the following proposition.

Proposition 2.4.19.

$$k[f_{11}, g_{11}] \mod J \cong k[f, g]$$

Proof. We have mod J matrices from the following form:

$$\overline{\overline{f}} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}, \overline{\overline{g}} = \begin{pmatrix} \lambda_2 & 0 & \dots & 0 \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

Then for any $H(\overline{\overline{f}}, \overline{\overline{g}}) \mod J$, we have

$$\overline{\overline{f}} = \begin{pmatrix} H(\lambda_1, \lambda_2) & 0 & \dots & 0 \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}.$$

□

Now we present an approach as follows. Consider $k(f, g)$. Let us extend the algebra of generic matrices by new matrix T, independent from all others. Consider conjugation of $k(f, g)$ by T, $Tk(f, g)T^{-1}$, and consider $\tilde{f} = TfT^{-1}$ and $\tilde{g} = TgT^{-1}$. By Corollary 2.4.17, we have

$$P(g_{11}, f_{11}) = 0 \mod J.$$

On the other hand, we have

$$k[f_{11}, g_{11}] \cap J = 0,$$

which means that

$$P(f_{11}, g_{11}) = 0 \mod J.$$

Put f_{11} and g_{11} be polynomial over commutative ring generated by all entries of $k[f, g]$ and T. Hence $\mathrm{Frac}(k(f, g))$ can be embedded into fractional field of rings of polynomials. According to Lüroth theorem, $\mathrm{Frac}(k(f, g))$ (hence $\mathrm{Frac}(C)$) is isomorphic to fields of rational functions in one variable.

This will not guarantee rationality of our field, and there are counter examples in this situation. However, this approach seems to be useful for highest term analysis.

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